Systems of Equations

Econ 6105, Fall 2024

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(SB Chapters 7, 8.4, 9.2-3, 15.3-4)

Introduction

Many economic environments are characterized by a system of multiple relationships between key variables

- E.g. Market price and quantity must satisfy both a Supply Curve and a Demand Curve

It is always important – and often not trivial – to know when/whether such a system has a solution, and when/whether it is unique.
This slide deck begins with linear algebra, which studies such environments when the relationships are linear.

- Builds intuition
- Relevant in some contexts, particularly in econometrics

The machinery of linear algebra turns out to also be useful for analyzing many non-linear settings.

- E.g. comparative statics essentially analyze tangent lines

Linear Systems in Matrix Form

Here are three ways people often represent a linear system of equations:

1. Long-form

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 = b_1$$

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 = b_2$$

2. Matrix form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \text{ or } Ax = b$$

3. Augmented matrix

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right)$$

Solving by Substitution

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 = b_1 \tag{1}$$

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 = b_2 \tag{2}$$

Equation 1 implies:

$$x_1 = \frac{b_1 - a_{12} \cdot x_2}{a_{11}}$$

Plugging this into Equation 2 yields:

$$a_{21} \cdot \frac{b_1 - a_{12} \cdot x_2}{a_{11}} + a_{22} \cdot x_2 = b_2,$$

or

$$x_2 = \frac{b_2 - a_{21} \cdot b_1 / a_{11}}{a_{22} - a_{21} \cdot a_{12} / a_{11}} = \frac{a_{11} \cdot b_2 - a_{21} \cdot b_1}{a_{11} \cdot a_{22} - a_{21} \cdot a_{21}}$$

So:

$$x_1 = \frac{b_1}{a_{11}} - a_{12} \cdot \frac{b_2 - a_{21} \cdot b_1 / a_{11}}{a_{11} \cdot a_{22} - a_{21} \cdot a_{12}} = \frac{a_{22} \cdot b_1 - a_{12} \cdot b_2}{a_{11} \cdot a_{22} - a_{21} \cdot a_{21}}$$

Reduced Row Echelon Form of a Matrix

A matrix is said to be in "row echelon form" if each row has more leading 0s – left to right – than the row before it.

$$\text{Yes: } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} ; \text{No: } \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note: The first non-zero entry in a row is called a "pivot."

A matrix is in "reduced row echelon form" if it is in row echcelon form and each pivot is 1, and all non-pivots are 0:

$$\text{No: } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} ; \text{Yes: } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Elementary Row Operations

We will soon use the augmented matrix to solve a system of linear equations.
The foundation for doing so is to note that there are 3 "elementary row operations" (EROs) such that if matrix A' results from performing an ERO on matrix A, they will yield the same solutions.
The operations are:

1. Interchanging two rows. E.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A' = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$
2. Multiplying a row by a constant $c \neq 0$.
   E.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A' = \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix}$
3. Changing a row by adding it to a multiple of another row.
   E.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A' = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

Solving With Augmented Matrix

Gaussian elimination can be done by shorthand using EROs to get the augmented matrix into reduced row echelon form.

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ 0 & a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} & b_2 - b_1 \cdot \frac{a_{21}}{a_{11}} \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{cc|c} a_{11} & 0 & b_1 - \frac{a_{12}}{a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}}}\left(b_2 - b_1 \cdot \frac{a_{21}}{a_{11}}\right) \\ 0 & a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} & b_2 - b_1 \cdot \frac{a_{21}}{a_{11}} \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}}\left(b_2 - b_1 \cdot \frac{a_{21}}{a_{11}}\right) \\ 0 & 1 & \frac{b_2 - b_1 \cdot \frac{a_{21}}{a_{11}}}{a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}}} \end{array}\right) \Rightarrow$$

The top entry of the final column is x_1, the bottom entry is x_2.
Takeaway: Matrices are slick ways of representing and working with systems of linear equations.

Row Reduction Numerical Example

$$x_1 - 3 \cdot x_3 = 4 \tag{3}$$

$$2 \cdot x_1 + 6 \cdot x_3 = 1 \tag{4}$$

$$-2 \cdot x_1 + 6 \cdot x_2 + 2 \cdot x_3 = 0 \tag{5}$$

Row Reduction Numerical Example

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$$-2 \cdot x_1 + 6 \cdot x_2 + 2 \cdot x_3 = 0 \tag{5}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 2 & 0 & 6 & 1 \\ -2 & 6 & 2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 0 & 12 & -7 \\ -2 & 6 & 2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ -2 & 6 & 2 & 0 \\ 0 & 0 & 12 & -7 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 6 & -4 & 8 \\ 0 & 0 & 12 & -7 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 6 & -4 & 8 \\ 0 & 0 & 1 & -7/12 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 - 7/4 \\ 0 & 6 & 0 & 8 - 7/3 \\ 0 & 0 & 1 & -7/12 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 - 7/4 \\ 0 & 1 & 0 & \frac{8-7/3}{6} \\ 0 & 0 & 1 & -7/12 \end{array}\right)$$

Systems Without a Unique Solution

Both examples we've done so far have been solved by a unique **x**.
Here are 3 examples that are not as "nice:"

1.
$$x_1 + x_2 = 0 \quad (6)$$
$$2 \cdot x_1 + 3 \cdot x_2 - x_3 = 2 \quad (7)$$

2.
$$x_1 + x_2 = 0 \quad (8)$$
$$2 \cdot x_1 + 3 \cdot x_2 = 2 \quad (9)$$
$$-x_1 + 4 \cdot x_2 = 4 \quad (10)$$

3.
$$x_1 + x_2 = 1 \quad (11)$$
$$2 \cdot x_1 + 2 \cdot x_2 = 2 \quad (12)$$

Systems Without a Unique Solution (2)

1.
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

Satisfied by any $\mathbf{x}$ with $-x_1 = x_2 = x_3 + 2$: ∞ solutions

2.
$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 2 \\ -1 & 4 & 4 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{array}\right)$$

Implies $0 \cdot x_1 + 0 \cdot x_2 = -6$: 0 solutions

3.
$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

Satisfied by any $\mathbf{x}$ with $x_1 = 1 - x_2$: ∞ solutions

Rank of a Matrix

The "rank" of a matrix is the number of non-zero rows in its row echelon form.

For the linear system $A\mathbf{x} = \mathbf{b}$, define $\hat{A}$ to be the augmented matrix $(A|\mathbf{b})$. Then:

1. A linear system has 0, 1, or ∞ solutions.
2. A solution to the system exists if and only if $\text{rank}(A) = \text{rank}(\hat{A})$
   - Note that $\text{rank}(A) \leq \text{rank}(\hat{A})$
3. A system with A is guaranteed to have a unique solution for any $\mathbf{b}$ if and only if $\text{rank}(A) = \#$ of A's rows $= \#$ of A's columns.

Examples With Rank

1. $$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

   rank(A)=rank($\hat{A}$) $= 2$, so a solution exists

2. $$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 2 \\ -1 & 4 & 4 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{array}\right)$$

   rank(A)$= 2 < 3 =$rank($\hat{A}$), so no solutions exist

3. $$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

   rank(A)=rank($\hat{A}$) $= 1$, so a solution exists

Examples With Rank

1. $$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

   rank(A)=2$< 3 = \#$ of A's columns, so no guarantee of uniqueness (∞ solutions)

2. $$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 2 \\ -1 & 4 & 4 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{array}\right)$$

   rank(A)$= 2 < 3 =$rank($\hat{A}$), so no solutions exist

3. $$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

   rank(A)=1$< 2 = \#$ of A's columns, so no guarantee of uniqueness (∞ solutions)

Intuition of Rank

$$A\mathbf{x} = \mathbf{b}$$

A's # of columns: # of variables to solve for (length of $\mathbf{x}$).

- Alternatively, # of free variables we have to satisfy equations

A's rank: # of independent pieces of information we have.

- Alternatively, # of equations we want to satisfy

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$$x_1 + x_2 = 0$$

$$2 \cdot x_1 + 3 \cdot x_2 - x_3 = 2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

Have 3 free variables but only need to satisfy 2 equations: ∞ ways to do this.

Intuition of Rank

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$$x_1 + x_2 = 0$$

$$2 \cdot x_1 + 3 \cdot x_2 = 2$$

$$-x_1 + 4 \cdot x_2 = 4$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{array}\right)$$

Have 2 free variables but need to satisfy 3 equations: 0 solutions unless very fortunate.

Intuition of Rank

$$A\mathbf{x} = \mathbf{b}$$

A's # of columns: # of variables to solve for (length of $\mathbf{x}$).

- Alternatively, # of free variables we have to satisfy equations

A's rank: # of independent pieces of information we have.

- Alternatively, # of equations we want to satisfy

$$x_1 + x_2 = 1$$

$$2 \cdot x_1 + 2 \cdot x_2 = 2$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

Have 2 free variables, and while we have 2 equations, they are redundant, so there's really only 1 equation to satisfy: ∞ solutions.

Intuition of Rank

$$A\mathbf{x} = \mathbf{b}$$

A's # of columns: # of variables to solve for (length of $\mathbf{x}$).

- Alternatively, # of free variables we have to satisfy equations

A's rank: # of independent pieces of information we have.

- Alternatively, # of equations we want to satisfy

$$x_1 - 3 \cdot x_3 = 4$$

$$2 \cdot x_1 + 6 \cdot x_3 = 1$$

$$-2 \cdot x_1 + 6 \cdot x_2 + 2 \cdot x_3 = 0$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 - 7/4 \\ 0 & 1 & 0 & \frac{8-7/3}{6} \\ 0 & 0 & 1 & -7/12 \end{array}\right)$$

rank(A)=rank($\hat{A}$)= # of A's columns $= 3$. Exactly 1 solution for any $\mathbf{b}$.

Rank and Determinant

$$A\mathbf{x} = \mathbf{b}$$

Consider a square 2x2 matrix, A. We know this will have a unique solution for any $\mathbf{b}$ if and only if rank(A)=2 ("full rank").
Row echelon form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} \end{bmatrix}$$

A is full rank if and only if $0 \neq a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

Rank and Determinant

$$A\mathbf{x} = \mathbf{b}$$

Consider a square 2x2 matrix, A. We know this will have a unique solution for any $\mathbf{b}$ if and only if rank(A)=2 ("full rank").
Row echelon form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} \end{bmatrix}$$

A is full rank if and only if $0 \neq a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = \det(A)$
Theorem: A square matrix has full rank if and only if its determinant is not 0.

The Inverse of a Square Matrix

The “identity matrix” is a diagonal matrix with 1s along the diagonal:

$$I \equiv \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ ... & ... & 1 & ... & ... \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & 0 & 1 \end{bmatrix}$$

The inverse of a square matrix A – denoted by A^{-1} – is defined such that:

$$A^{-1}A = I \tag{13}$$

Computing the Inverse of a 2x2 Matrix

Take $(A|I)$ and turn it into $(I|B)$. $B = A^{-1}$.

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ 0 & a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} & -\frac{a_{21}}{a_{11}} & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} a_{11} & 0 & 1 + \frac{a_{12} \cdot a_{21}}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} & -\frac{a_{12}}{a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}}} \\ 0 & a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} & -\frac{a_{21}}{a_{11}} & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{11} \cdot a_{22}}{a_{11} \cdot (a_{11} \cdot a_{22} - a_{12} \cdot a_{21})} & -\frac{a_{12}}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} \\ 0 & 1 & -\frac{a_{21}}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} & \frac{1}{a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}}} \end{array}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Computing the Inverse of a 2x2 Matrix

Take $(A|I)$ and turn it into $(I|B)$. $B = A^{-1}$.

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ 0 & a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} & -\frac{a_{21}}{a_{11}} & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} a_{11} & 0 & 1 + \frac{a_{12} \cdot a_{21}}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} & -\frac{a_{12}}{a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}}} \\ 0 & a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} & -\frac{a_{21}}{a_{11}} & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{11} \cdot a_{22}}{a_{11} \cdot (a_{11} \cdot a_{22} - a_{12} \cdot a_{21})} & -\frac{a_{12}}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} \\ 0 & 1 & -\frac{a_{21}}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} & \frac{1}{a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}}} \end{array}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

A Bunch of Theorems

$$A\mathbf{x} = \mathbf{b}$$

The following statements are equivalent if A is *nxn*.

1. $\det(A) \neq 0$
2. A is invertible
3. A has full rank
4. The system has a unique solution, with:

$$\mathbf{x} = A^{-1}\mathbf{b} \tag{14}$$

Cramer's Rule

$$A\mathbf{x} = \mathbf{b}$$

When A is nxn and $\mathbf{x}$ is its solution, we have:

$$x_i = \frac{\det(B_i)}{\det(A)}, \tag{15}$$

where B_i is found by replacing the i^{th} column of A with $\mathbf{b}$.

Showing Cramer's Rule for 2x2

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- $\det(A) = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$
- $B_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$
  - $\det(B_1) = b_1 \cdot a_{22} - b_2 \cdot a_{12}$
- $B_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$
  - $\det(B_2) = b_2 \cdot a_{11} - b_1 \cdot a_{21}$

$$x_1 = \frac{b_1 \cdot a_{22} - b_2 \cdot a_{12}}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}}, x_2 = \frac{b_2 \cdot a_{11} - b_1 \cdot a_{21}}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}}$$

Linearizing a System

Consider the following system, where f_1,...,f_m: $R^n \to R^1$ are differentiable:

$$f_1(x_1, ..., x_n) = c_1$$

$$...$$

$$f_m(x_1, ..., x_n) = c_m$$

Suppose $\mathbf{x}^* \equiv (x_1^*, ..., x_n^*)$ satisfies the system. Take total derivatives to get an arbitrarily good approximation of the non-linear system at $\mathbf{x}^*$:

$$\frac{\partial f_1}{\partial x_1} \cdot dx_1 + ... + \frac{\partial f_1}{\partial x_n} \cdot dx_n = 0$$

$$...$$

$$\frac{\partial f_m}{\partial x_1} \cdot dx_1 + ... + \frac{\partial f_m}{\partial x_n} \cdot dx_n = 0$$

where all partial derivatives are evaluated at $\mathbf{x}^*$.

Example of Linearization

$$p \cdot u_1(x_1^*, x_2^*) - u_2(x_1^*, x_2^*) = 0 \tag{16}$$

$$x_1^* + p \cdot x_2^* = I \tag{17}$$

1. Equation 16 has a non-linear utility function in it
2. If we consider p and I to be variables (which they are...), then both equations involve products of variables, i.e. non-linear

The linearization of this system is:

$$(p \cdot u_{11} - u_{12}) \cdot dx_1 + (p \cdot u_{12} - u_{22}) \cdot dx_2 + u_1 \cdot dp = 0 \tag{18}$$

$$1 \cdot dx_1 + p \cdot dx_2 + x_2^* \cdot dp - 1 \cdot dI = 0 \tag{19}$$

Matrix Representation of Linearized System

$$(p \cdot u_{11} - u_{12}) \cdot dx_1 + (p \cdot u_{12} - u_{22}) \cdot dx_2 + u_1 \cdot dp = 0 \tag{18}$$

$$1 \cdot dx_1 + p \cdot dx_2 + x_2^* \cdot dp - 1 \cdot dI = 0 \tag{19}$$

$$A\mathbf{z} = \mathbf{0}, \tag{20}$$

where $A \equiv \begin{bmatrix} p \cdot u_{11} - u_{12} & p \cdot u_{12} - u_{22} & u_1 & 0 \\ 1 & p & x_2^* & -1 \end{bmatrix}$ is 2x4, and

$$\mathbf{z} \equiv \begin{bmatrix} dx_1 \\ dx_2 \\ dp \\ dI \end{bmatrix} \text{ is } 4x1.$$

Endogenous Versus Exogenous Variables

We can put more structure on this by categorizing the variables:

1. Exogenous: inputs the modeler can change (here, likely p and I)
2. Endogenous: outputs the model explains (here, likely x_1^* and x_2^*)

Restrict $\mathbf{z}$ to endogenous variables:

$$A\mathbf{z} = \mathbf{b}, \tag{21}$$

where $A \equiv \begin{bmatrix} p \cdot u_{11} - u_{12} & p \cdot u_{12} - u_{22} \\ 1 & p \end{bmatrix}$ is 2x2,

$\mathbf{z} \equiv \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$ is 2x1, and $\mathbf{b} \equiv \begin{bmatrix} -u_1 \cdot dp \\ dI - x_2^* \cdot dp \end{bmatrix}$ is 2x1.

Very important that we have the same number of equations as endogenous variables. Why?

A Comparative Static

$$A\mathbf{z} = \mathbf{b},$$

where $A \equiv \begin{bmatrix} p \cdot u_{11} - u_{12} & p \cdot u_{12} - u_{22} \\ 1 & p \end{bmatrix}$ is $2x2$,

$\mathbf{z} \equiv \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$ is $2x1$, and $\mathbf{b} \equiv \begin{bmatrix} -u_1 \cdot dp \\ dI - x_2^* \cdot dp \end{bmatrix}$ is $2x1$.

Can solve for dx_2, e.g. using Cramer's Rule:

$$dx_2 = \frac{\det \begin{pmatrix} p \cdot u_{11} - u_{12} & -u_1 \cdot dp \\ 1 & dI - x_2^* \cdot dp \end{pmatrix}}{\det(A)} = \frac{u_1 \cdot dp + dI \cdot (p \cdot u_{11} - u_{12}) - x_2^* \cdot (p \cdot u_{11} - u_{12}) \cdot dp}{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}}$$

If $dI = 0$, i.e. if we want to see the impact of a change in p, holding income constant:

$$\frac{dx_2}{dp} = \frac{u_1 - x_2^* \cdot (p \cdot u_{11} - u_{12})}{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}},$$

the comparative static we've seen before.

The Implicit Function Theorem, Revisited

Consider differentiable function $G_1 : R^{n+1} \to R^1$ and $\mathbf{x} \in R^n$:

$$G_1\big(\mathbf{x}, y_1^*(\mathbf{x})\big) = c_1.$$

If:

$$\frac{\partial G_1}{\partial y_1^*} \neq 0 \tag{22}$$

when evaluated at $\mathbf{x}$, then at $\mathbf{x}$, differentiable $y_1^*(\mathbf{x})$ exists, and

$$\frac{\partial y_1^*}{\partial x_i} = -\frac{\frac{\partial G_1}{\partial x_i}}{\frac{\partial G_1}{\partial y_1^*}}$$

The Implicit Function Theorem, Revisited

Consider differentiable functions $G_1, ..., G_m : R^{n+m} \to R^1$ and $\mathbf{x} \in R^n$:

$$G_1\big(\mathbf{x}, y_1^*(\mathbf{x}), ..., y_m^*(\mathbf{x})\big) = c_1.$$

$$...$$

$$G_m\big(\mathbf{x}, y_1^*(\mathbf{x}), ..., y_m^*(\mathbf{x})\big) = c_m.$$

If:

$$\det \begin{pmatrix} \frac{\partial G_1}{\partial y_1^*} & ... & \frac{\partial G_1}{\partial y_m^*} \\ ... & ... & ... \\ \frac{\partial G_m}{\partial y_1^*} & ... & \frac{\partial G_m}{\partial y_m^*} \end{pmatrix} \neq 0 \tag{22}$$

when evaluated at $\mathbf{x}$, then at $\mathbf{x}$, differentiable $y_1^*(\mathbf{x})$,...,$y_m^*(\mathbf{x})$ exists, and $\frac{\partial y_k^*}{\partial x_h}$ can be found by linearizing the system, setting $dx_h = 1$ and $dx_{j \neq h} = 0$ and solving the system (e.g. by inverting the matrix above or using Cramer's Rule).

The Implicit Function Theorem, Revisited (2)

Condition 22 says we need as many (independent) equations as unknowns (ys) – that gets a non-zero determinant.

$$A\mathbf{z} = \mathbf{b},$$

where $A \equiv \begin{bmatrix} p \cdot u_{11} - u_{12} & p \cdot u_{12} - u_{22} \\ 1 & p \end{bmatrix}$ is $2x2$,
$\mathbf{z} \equiv \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$ is $2x1$, and $\mathbf{b} \equiv \begin{bmatrix} -u_1 \cdot dp \\ dI - x_2^* \cdot dp \end{bmatrix}$ is $2x1$.
Can solve for dx_2, e.g. using Cramer's Rule:

$$dx_2 = \frac{\det \begin{pmatrix} p \cdot u_{11} - u_{12} & -u_1 \cdot dp \\ 1 & dI - x_2^* \cdot dp \end{pmatrix}}{\det(A)}$$

If $\det(A) = 0$, the dx_2 will be undefined – comparative static, $\frac{dx_2}{dp}$, will not exist.
In fact, none of the comparative statics ($\frac{dx_1}{dp}, \frac{dx_1}{dI}, \frac{dx_2}{dI}$) will exist.

The Other Comparative Statics

$$A\mathbf{z} = \mathbf{b},$$

where $A \equiv \begin{bmatrix} p \cdot u_{11} - u_{12} & p \cdot u_{12} - u_{22} \\ 1 & p \end{bmatrix}$ is $2x2$,

$\mathbf{z} \equiv \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$ is $2x1$, and $\mathbf{b} \equiv \begin{bmatrix} -u_1 \cdot dp \\ dI - x_2^* \cdot dp \end{bmatrix}$ is $2x1$.

- $\frac{dx_2}{dp} = \frac{u_1 + x_2^* \cdot (u_{12} - p \cdot u_{11})}{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}}$
- $\frac{dx_2}{dI} =$
- $\frac{dx_1}{dp} =$
- $\frac{dx_1}{dI} =$

The Other Comparative Statics

$$A\mathbf{z} = \mathbf{b},$$

where $A \equiv \begin{bmatrix} p \cdot u_{11} - u_{12} & p \cdot u_{12} - u_{22} \\ 1 & p \end{bmatrix}$ is 2x2,

$\mathbf{z} \equiv \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$ is 2x1, and $\mathbf{b} \equiv \begin{bmatrix} -u_1 \cdot dp \\ dI - x_2^* \cdot dp \end{bmatrix}$ is 2x1.

- $\frac{dx_2}{dp} = \frac{u_1 + x_2^* \cdot (u_{12} - p \cdot u_{11})}{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}}$
- $\frac{dx_2}{dI} = \frac{p \cdot u_{11} - u_{12}}{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}}$
- $\frac{dx_1}{dp} = \frac{x_2^* \cdot (p \cdot u_{12} - u_{22}) - p \cdot u_1}{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}}$
- $\frac{dx_1}{dI} = -\frac{p \cdot u_{12} - u_{22}}{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}}$