Long-run Growth

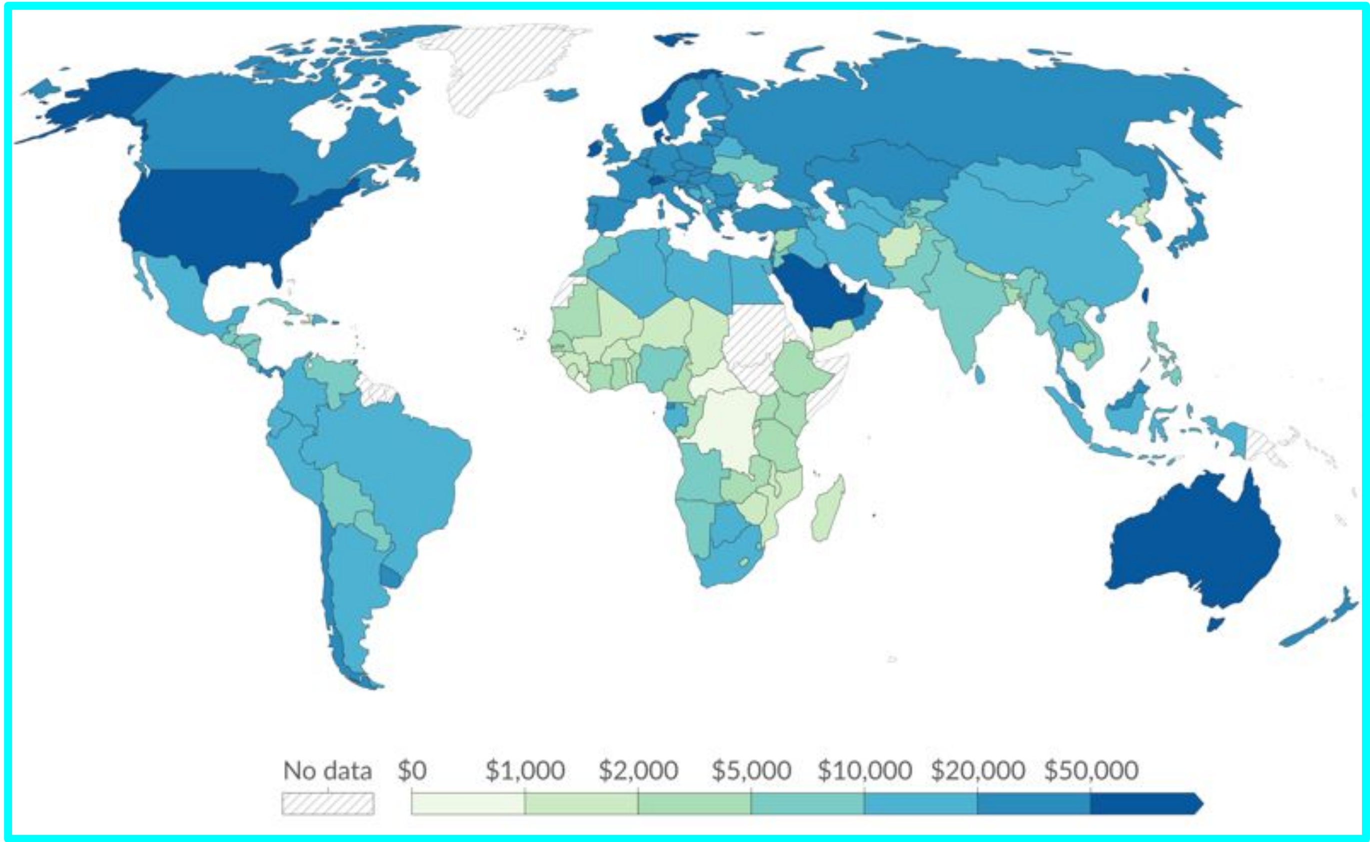
Spring 2026
Econ 256, College of the Holy Cross
Prof. Josh Abel

Chapters 9-11 and Khan (2009)

Facts about Long-run Growth/Development

Average Income by Country, 2022

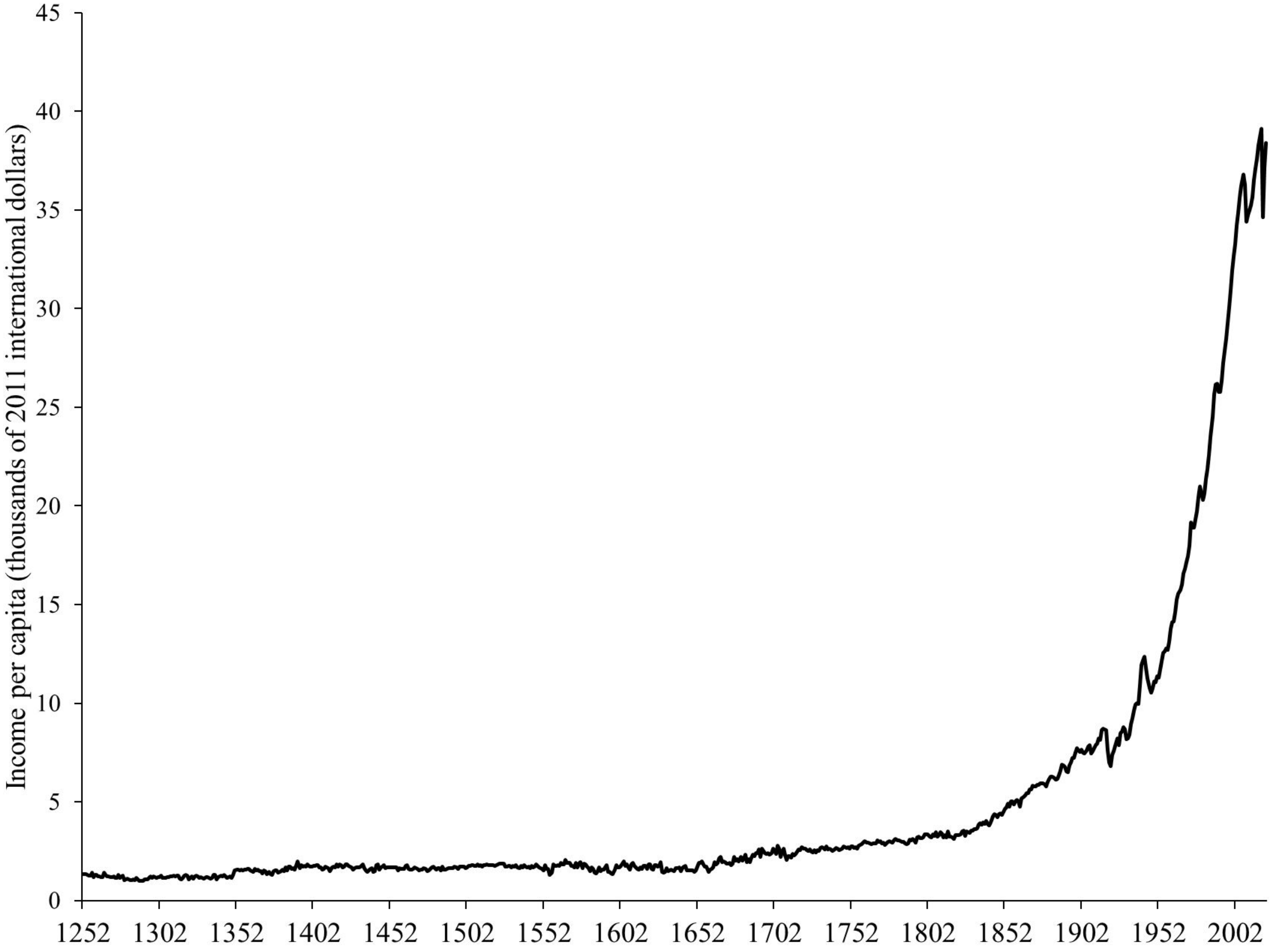
- Economists usually use the word "development" to mean the set of issues surrounding why some places are rich and others are poor
    - i.e. looking across space



Sources:
[1] Maddison Project Database
[2] Our World in Data

UK saw explosive (exponential) growth starting in 1700s

- Economists usually use the word “growth” to mean the set of issues surrounding why living standards have generally risen sharply over the past few centuries
    - i.e. looking across time



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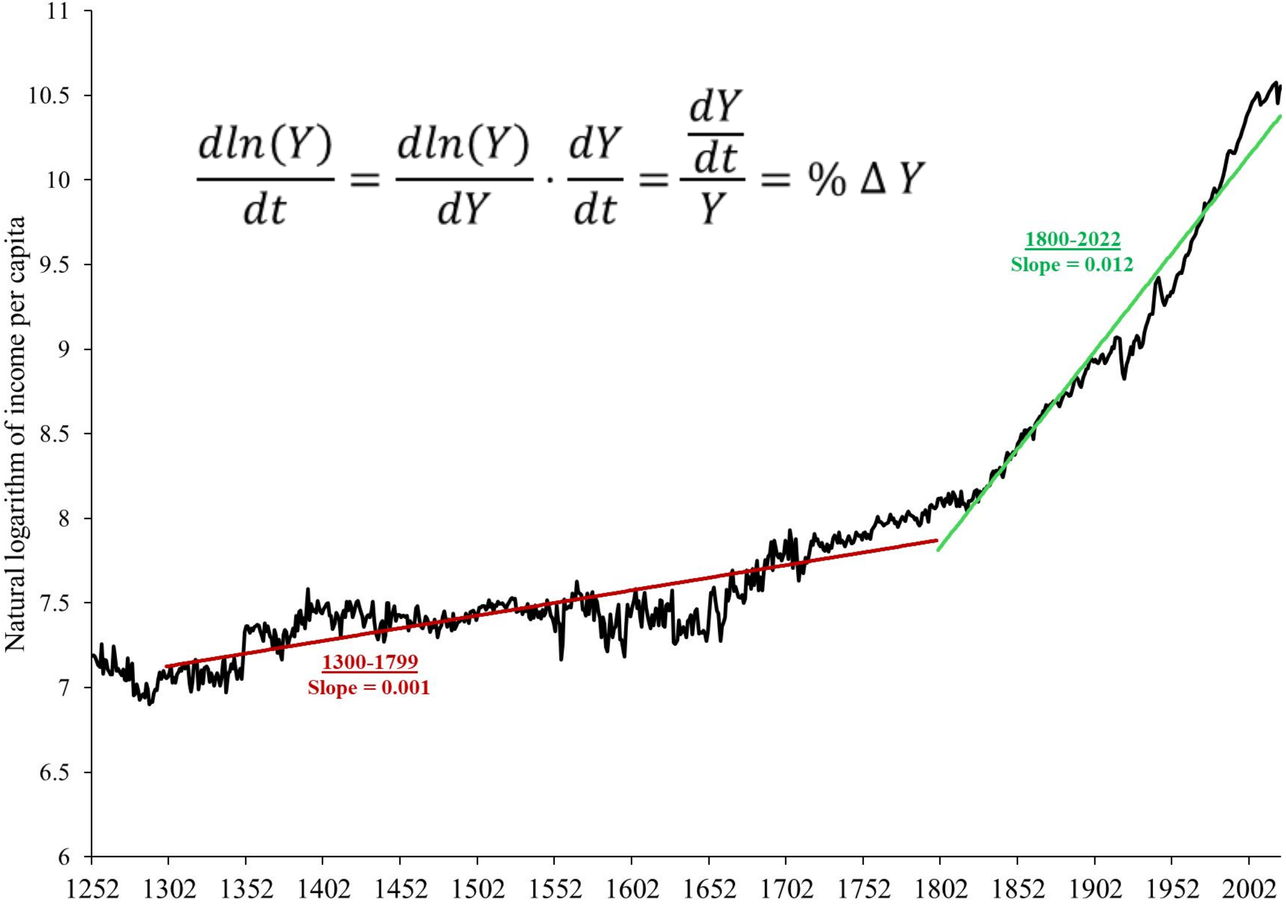
Logarithms are useful for looking at exponential growth

- Output per capita grew 0.1% per year from 1300-1799 in UK
- This rose dramatically to 1.2% per year from 1800-2022



$$\frac{d\ln(Y)}{dt} = \frac{d\ln(Y)}{dY} \cdot \frac{dY}{dt} = \frac{\frac{dY}{dt}}{Y} = \% \Delta Y$$

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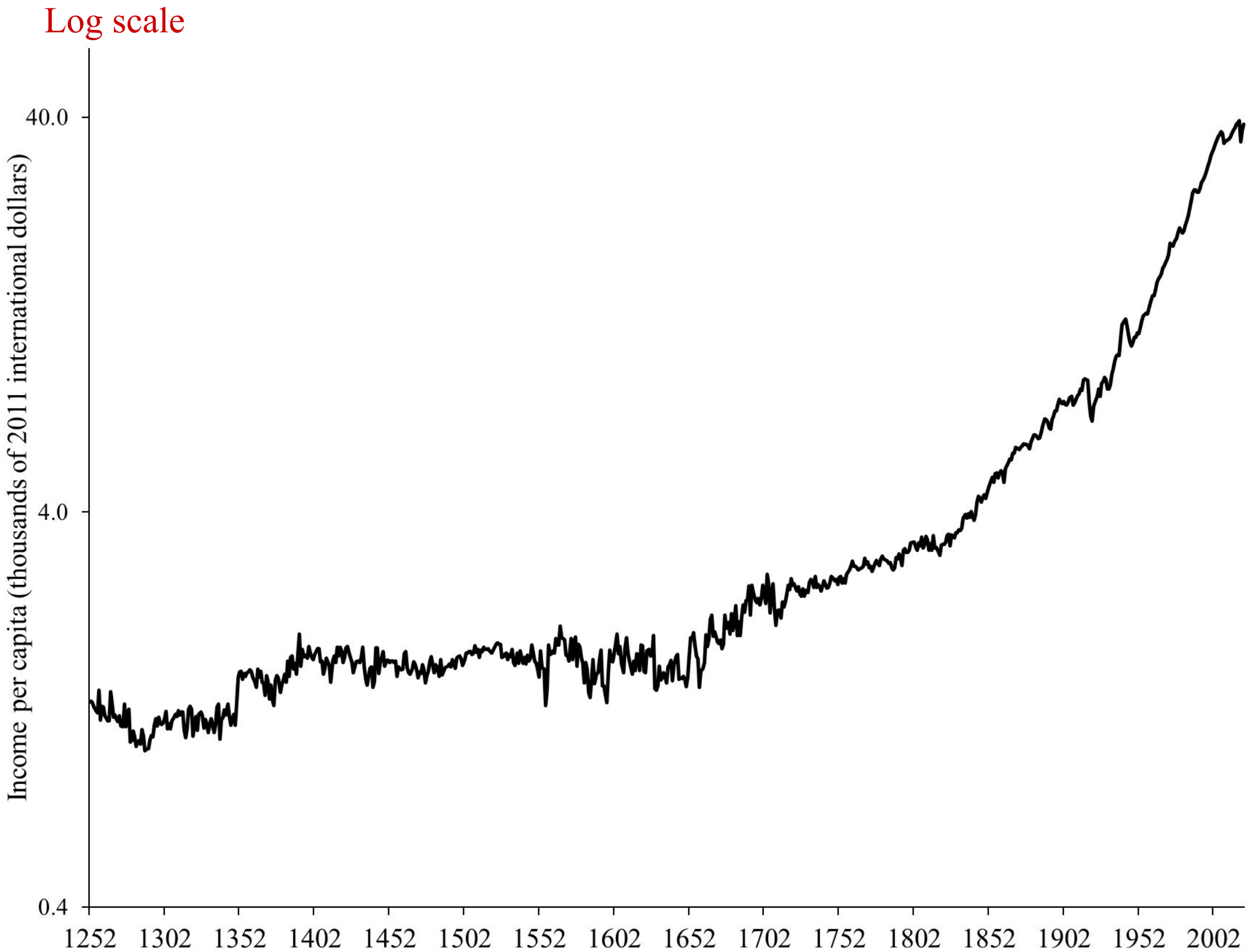

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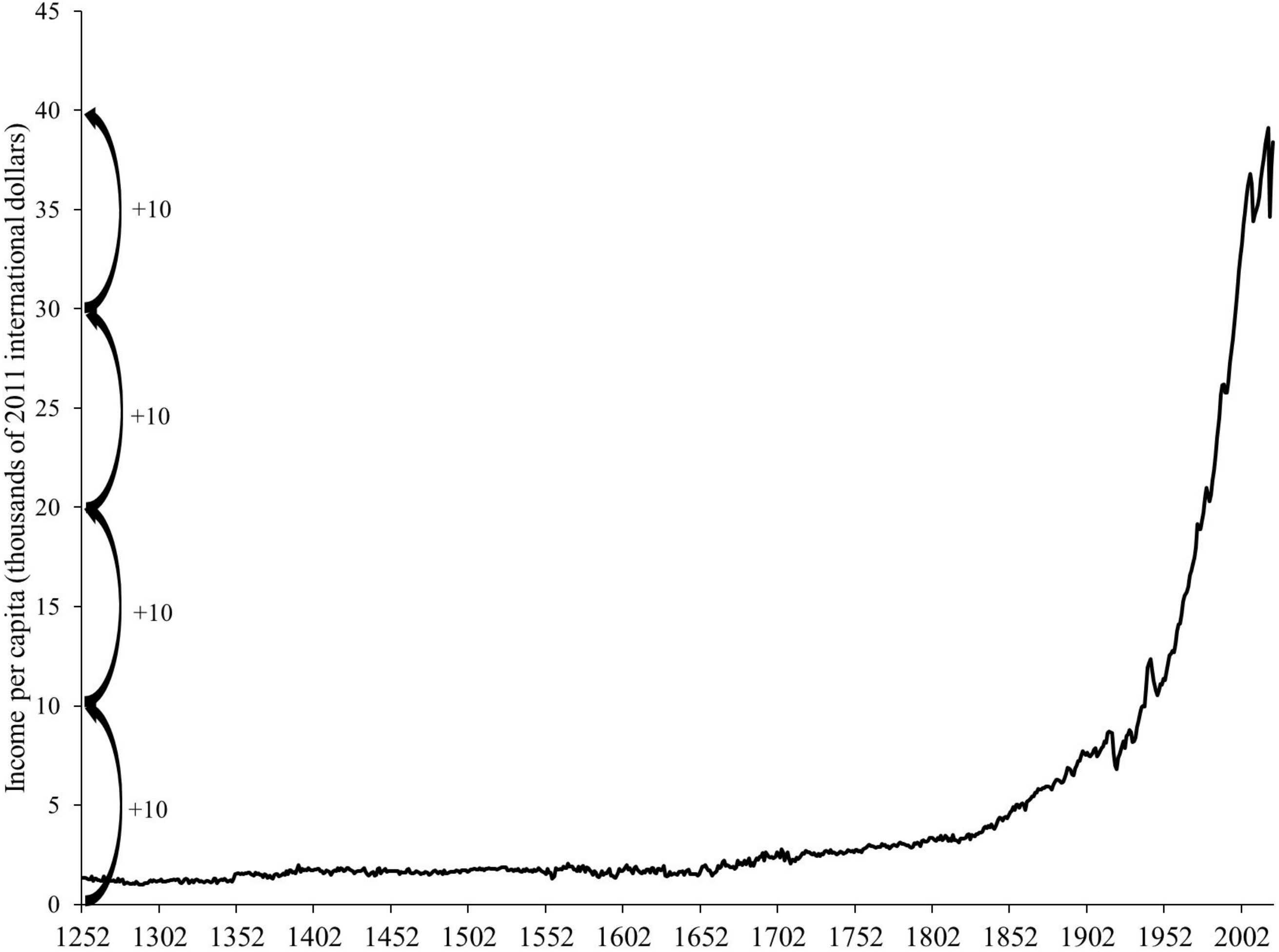

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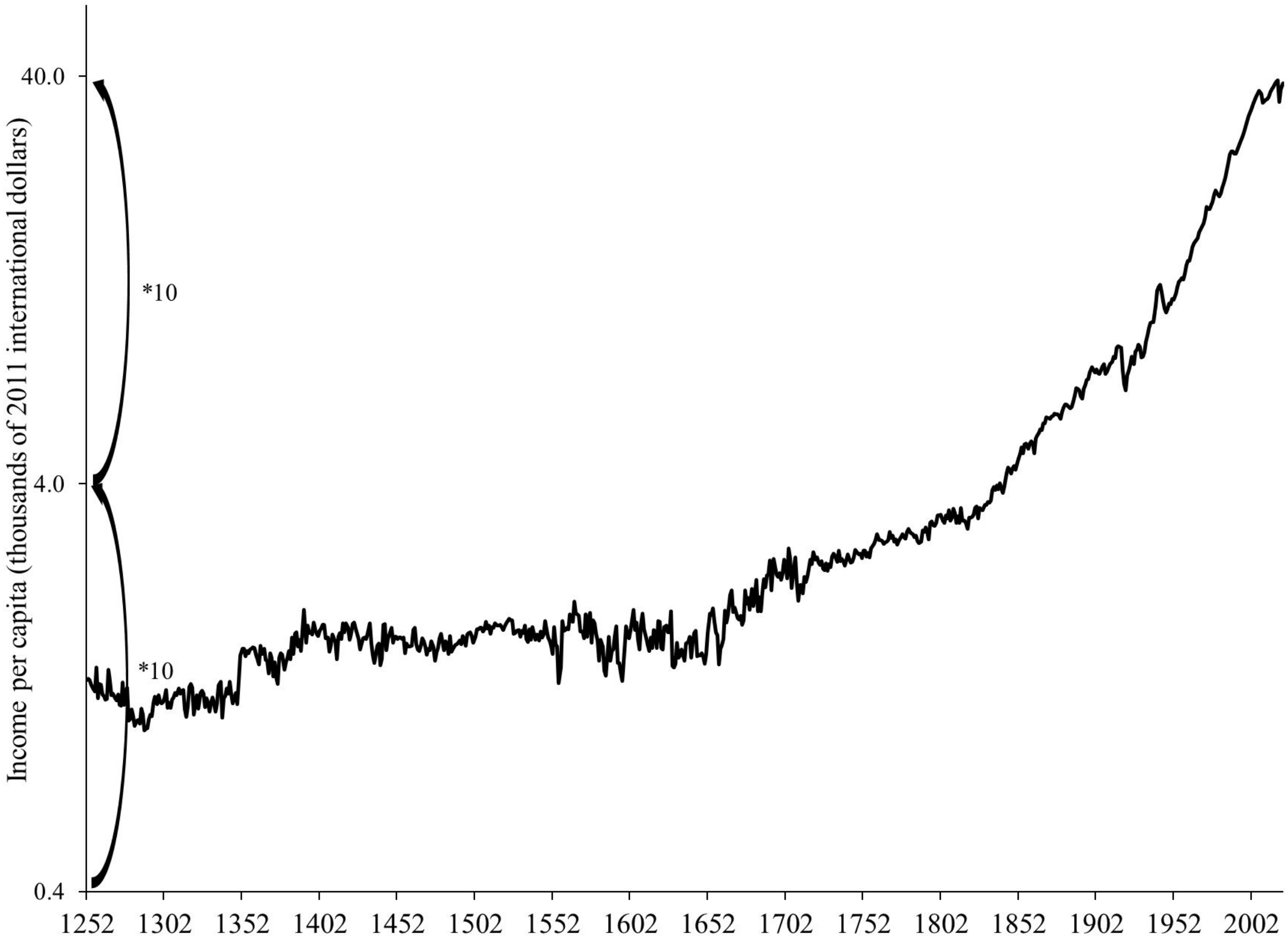

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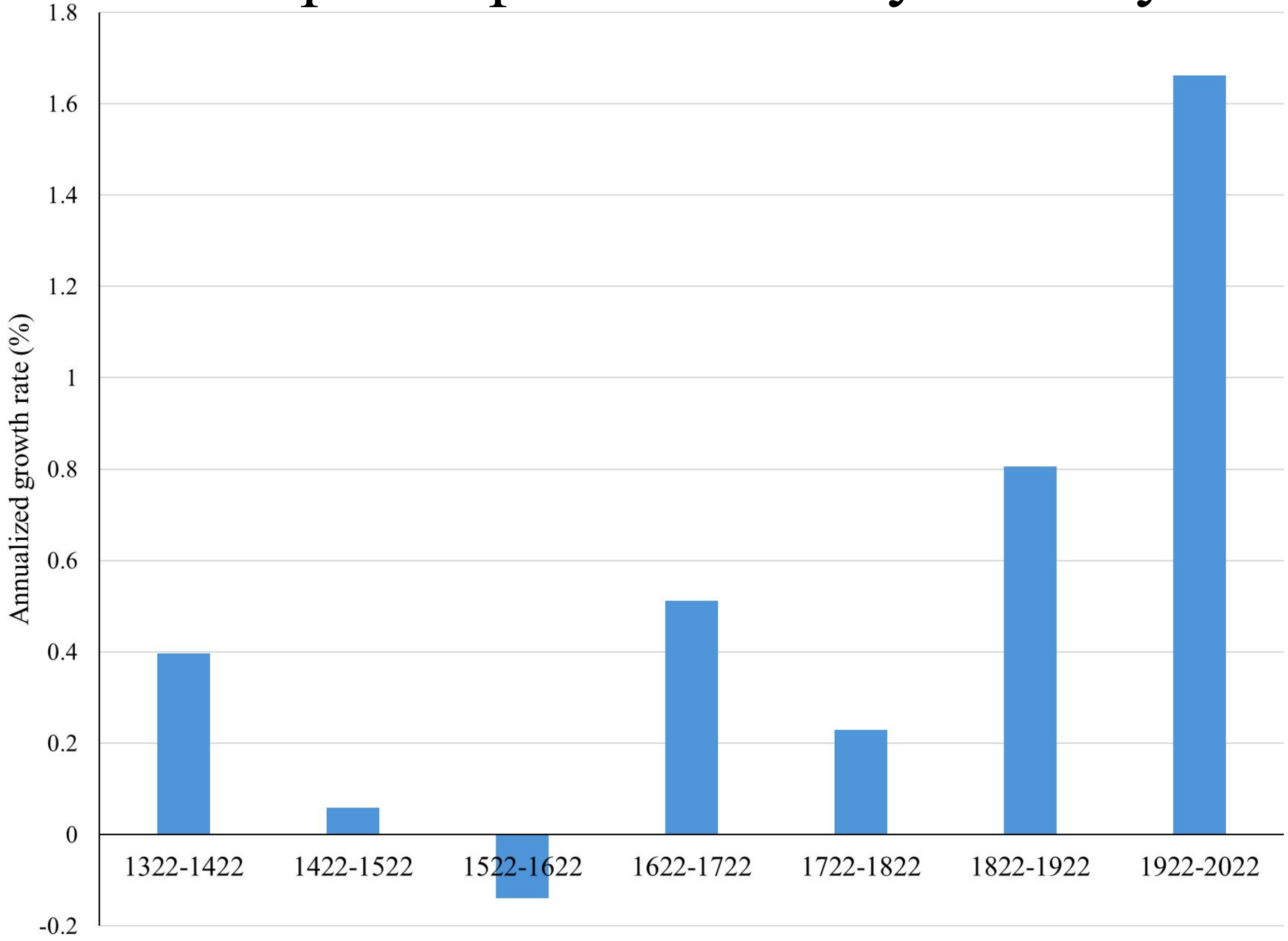
UK Income per Capita Growth by Century



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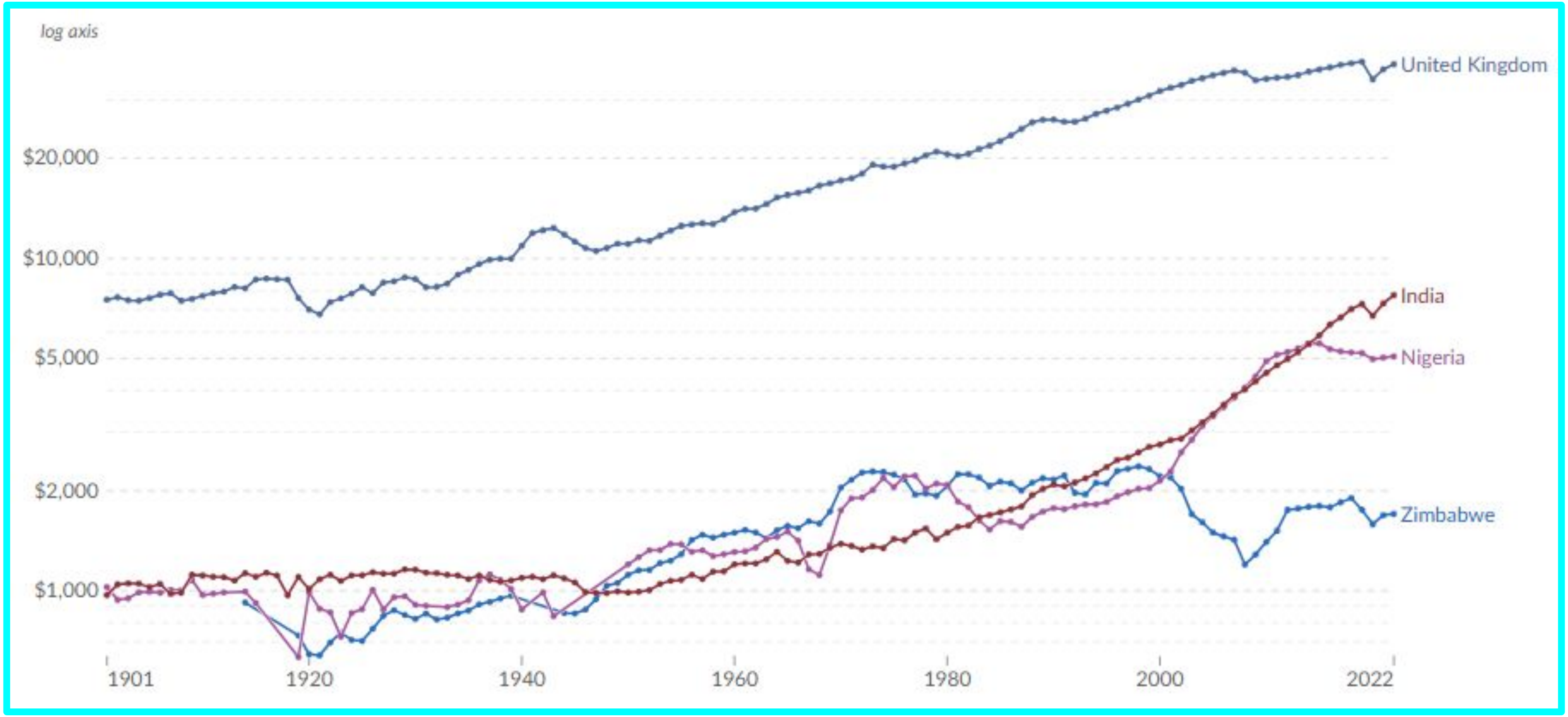
It’s been a mixed bag for the UK’s former colonies



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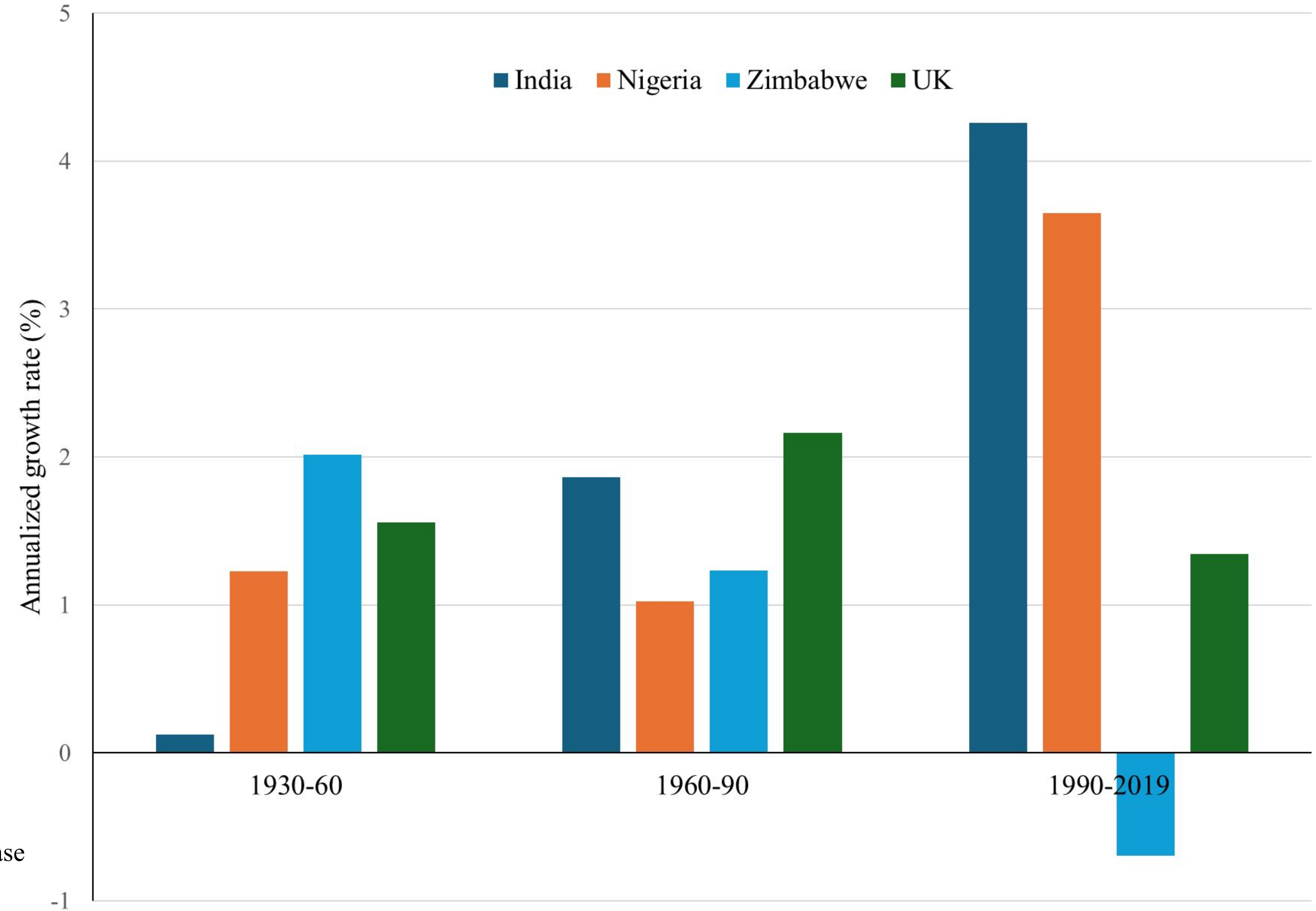
It's been a mixed bag for UK's former colonies



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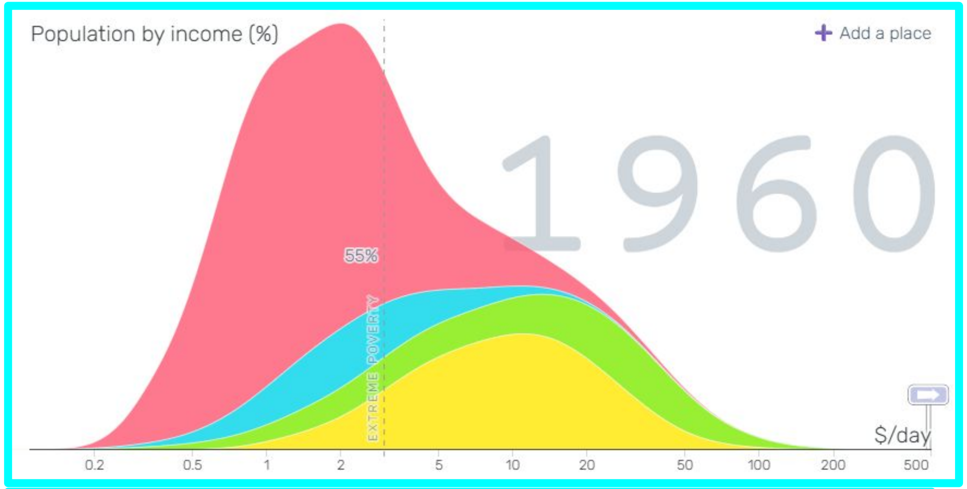
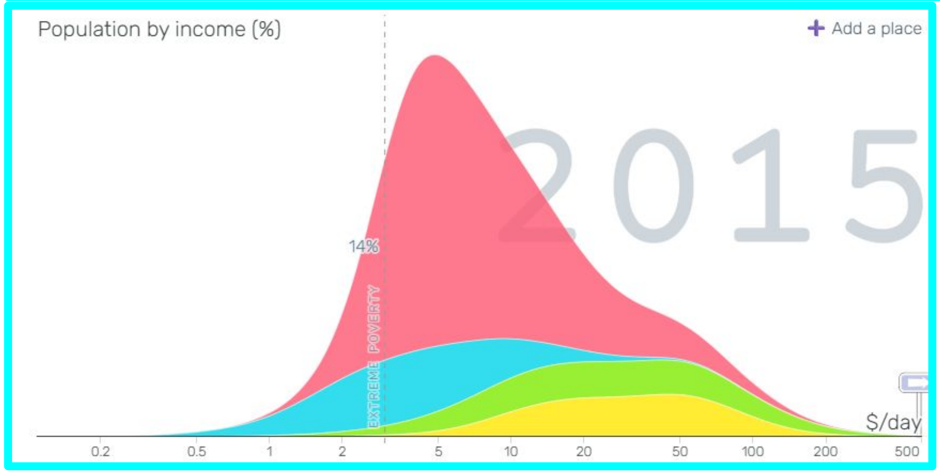
Income Distribution Across Time and Space





Source: www.gapfinder.org

The Solow Model

Goals

- One clear potential driver of output is capital:
  - If we've invested in more machines, we can produce more output
- If this is the key to higher output – and higher output growth – then there is a clear policy solution: invest more!
  - Whether through foreign aid or their own savings, poor countries could raise output by accumulating more capital
- The Solow Model is the canonical framework for thinking about this issue
  - It is simple to the point of elegance, yet allows for a rich understanding of the dynamics of capital, investment, output, and consumption
  - As a result, it is the mental model most economists use to think about long-run growth and development
  - As importantly, it is straightforward to apply to the data and draw real-world takeaways
    - Unfortunately, the main takeaway is that growth and development are really hard to explain!

Key question: can growth and development be explained by capital accumulation?

Setup

- Closed economy with no government spending

$$Y_t = C_t + I_t$$

- We will simplify behavior:
  - Rather than solving for optimal consumption, assume a constant savings rate of s:

$$C_t = (1\text{-}s)*Y_t$$

$$I_t = s*Y_t$$

Production Function

$$Y_t = F(K_t, E_t * L_t)$$

- Output is a function of capital (K), labor (L), and "labor effectiveness" (E) – for now just assume $E_t = 1$
- Convenient to assume F has "constant returns to scale"

$$F(x \cdot K, x \cdot L) = x \cdot F(K, L)$$

  - Doubling inputs doubles the output. Implies:

$$y_t \equiv \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) \equiv f(k_t)$$

- Critical (and reasonable) to assume production function F has "decreasing marginal product"
  - Each unit of capital produces less than the unit before it
  - Just like utility function, we have concavity:
    1. $f'(k) > 0$
    2. $f''(k) < 0$

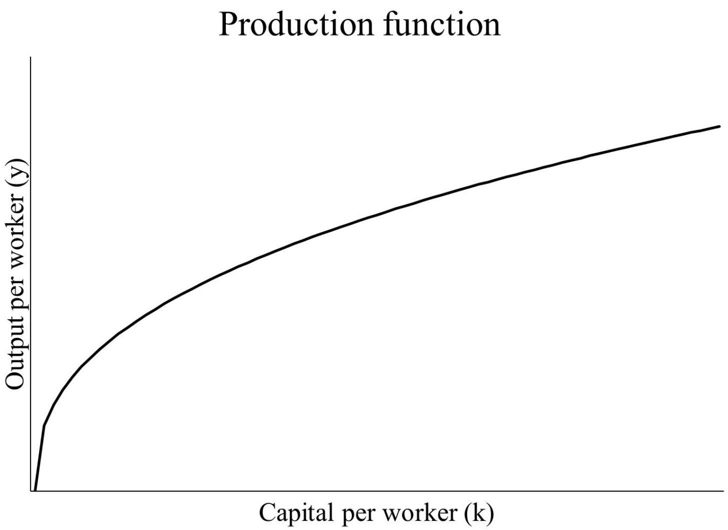
Decreasing marginal product

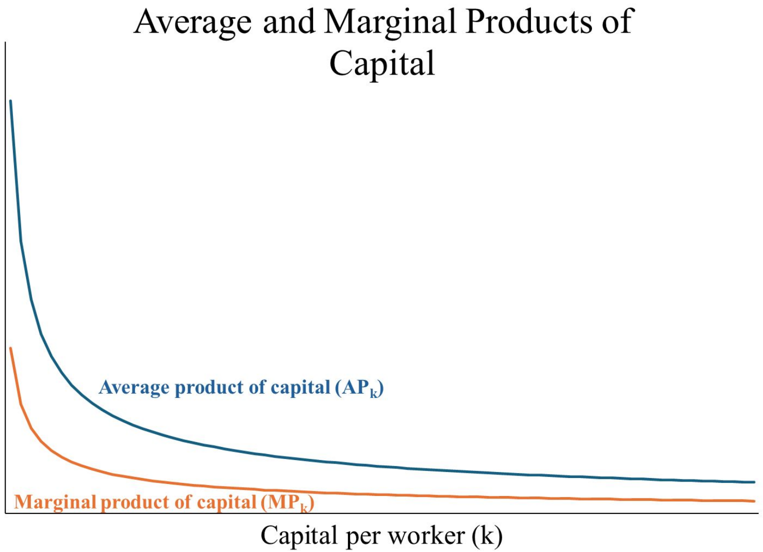
Ex)

$$y = k^{1-\alpha}$$

$$MP_k = (1-\alpha)/k^{\alpha}$$

$$AP_k = 1/k^{\alpha}$$





Capital's Law of Motion

- Capital changes for 2 reasons:
  1. Investment, I_t, increases capital
  2. Depreciation, $\boldsymbol{\delta}*K_t$, lowers capital
- Capital's Law of Motion
  1. Discrete time:

$$K_{t+1} = (1-\boldsymbol{\delta})*K_t + I_t \qquad \text{or} \qquad K_{t+1} - K_t = I_t - \boldsymbol{\delta}*K_t$$

  2. Continuous time:

$$\dot{K}_t \equiv \frac{dK_t}{dt} = I_t - \delta \cdot K_t$$

The Solow Model

- Exogenous variables:
  1. Initial conditions: K_0, L_0, E_0
  2. Growth rates of labor and technology, g_L and g_E
     - This implies that the time paths of labor and technology, L_t and E_t for $t > 0$, are exogenous, too
     - To begin, we will assume $g_L = g_E = 0$
  3. Depreciation: $\boldsymbol{\delta}$
  4. Savings rate: s – this is the main policy lever we will consider
- Endogenous variables:
  1. **Path of capital**, K_t for $t > 0$
  2. Resulting paths of output and consumption, Y_t and C_t for $t > 0$
  3. **Steady state level of capital**, K_{ss} such that $\dot{K}_t = 0$
  4. Resulting steady state levels of output and consumption, Y_{ss} and C_{ss}

Break-even Investment

$$\dot{K}_t \equiv \frac{dK_t}{dt} = I_t - \delta \cdot K_t$$

- Because L is constant, we can divide through by L and get:

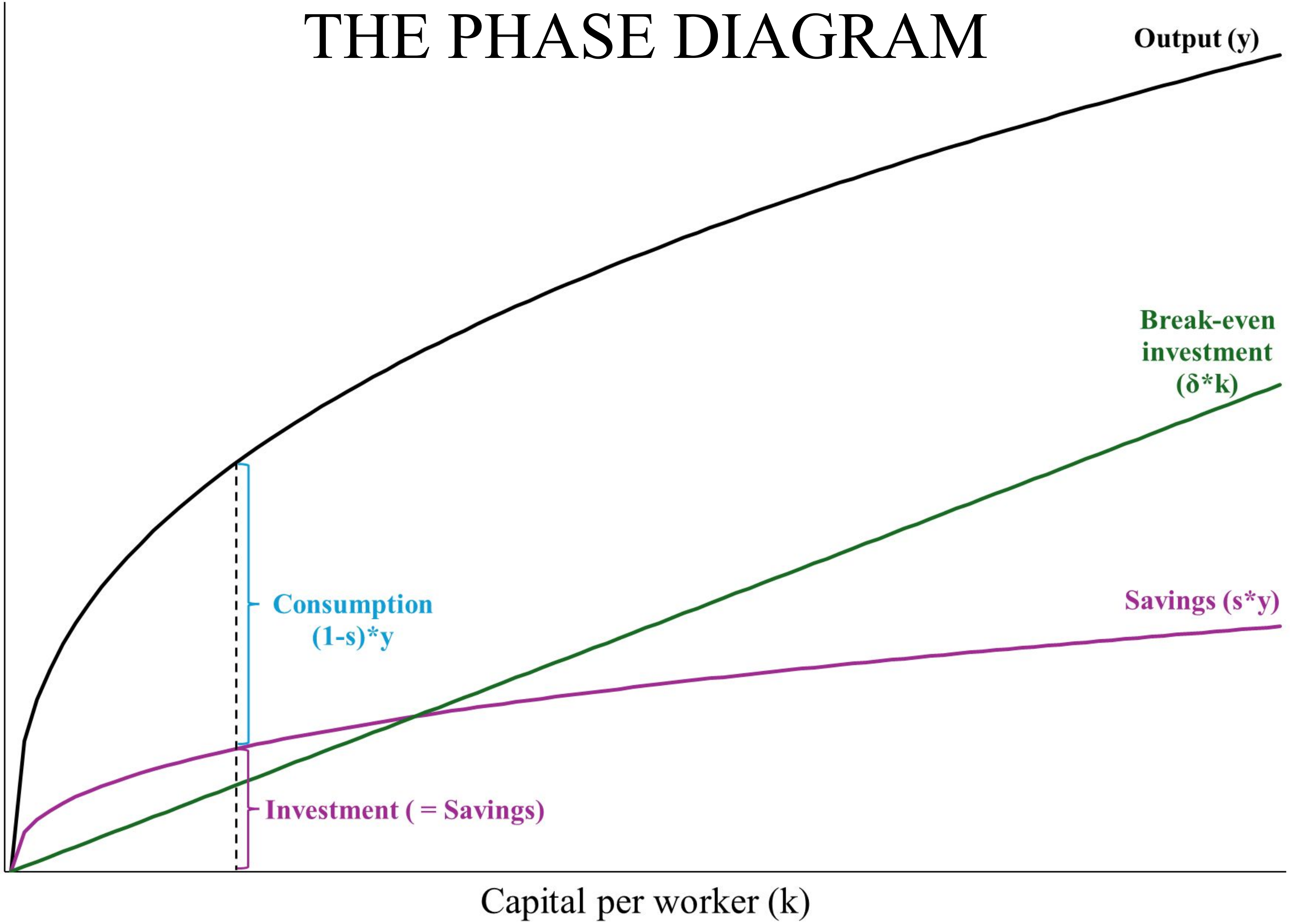
$$\dot{k}_t = s \cdot f(k_t) - \delta \cdot k_t$$

- So capital is constant when:

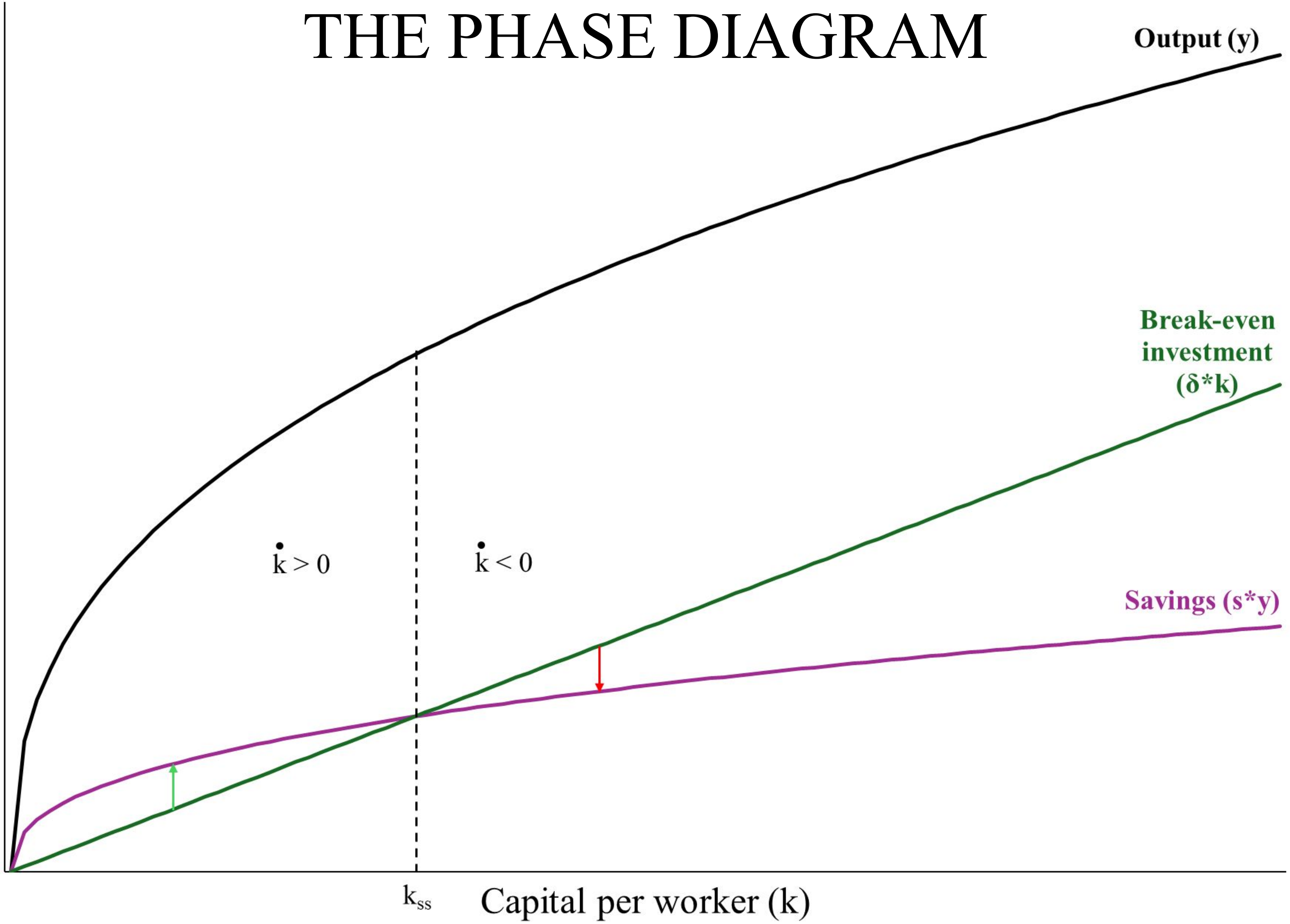
$$s \cdot f(k_t) = \delta \cdot k_t$$

- Because capital has decreasing marginal product:
  1. $\dot{k}_t > 0$ when k_t is small
     - Depreciation is low and capital is quite productive on average
  2. $\dot{k}_t < 0$ when k_t is large
     - Depreciation is high and capital is not very productive on average

THE PHASE DIAGRAM



THE PHASE DIAGRAM



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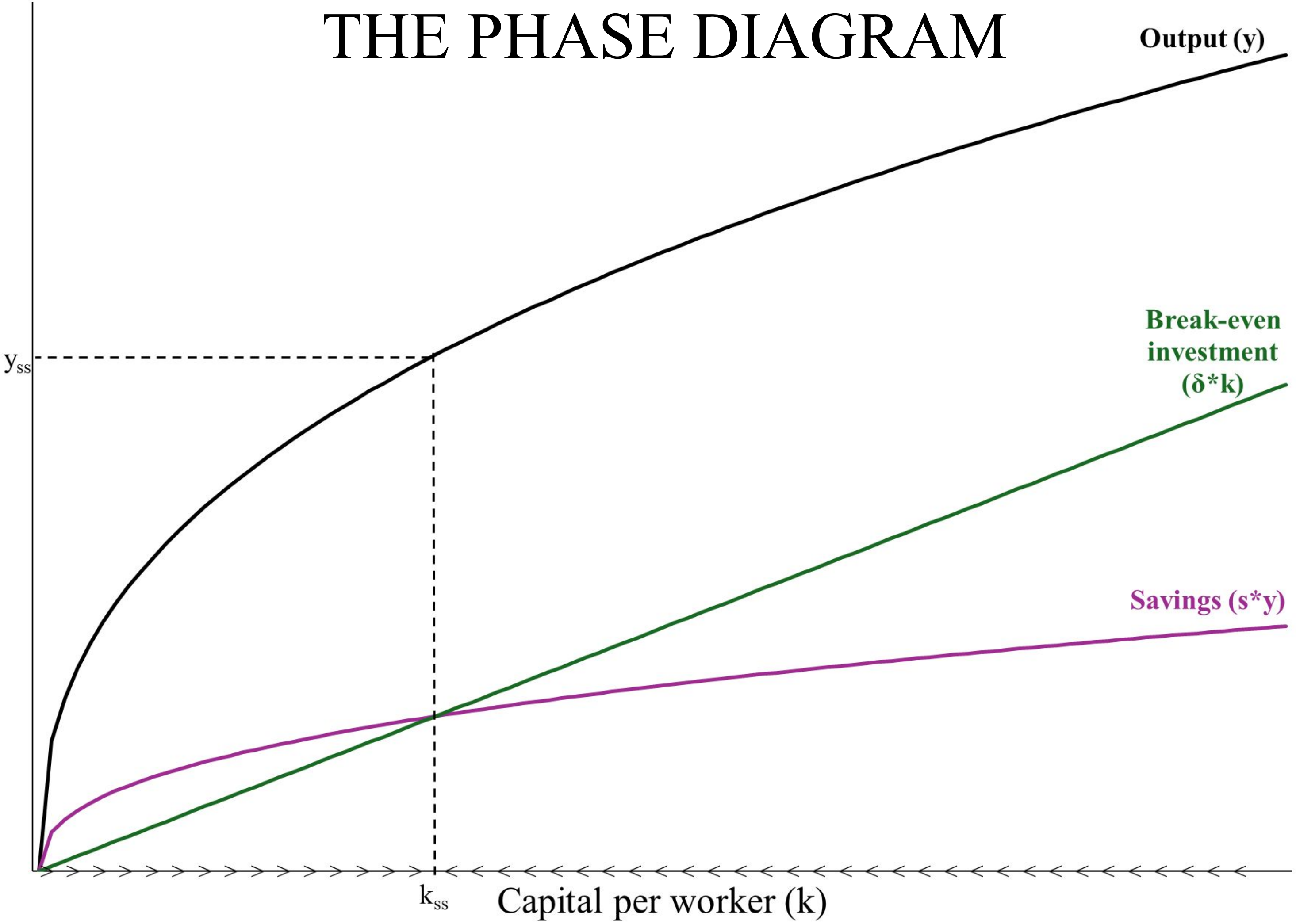
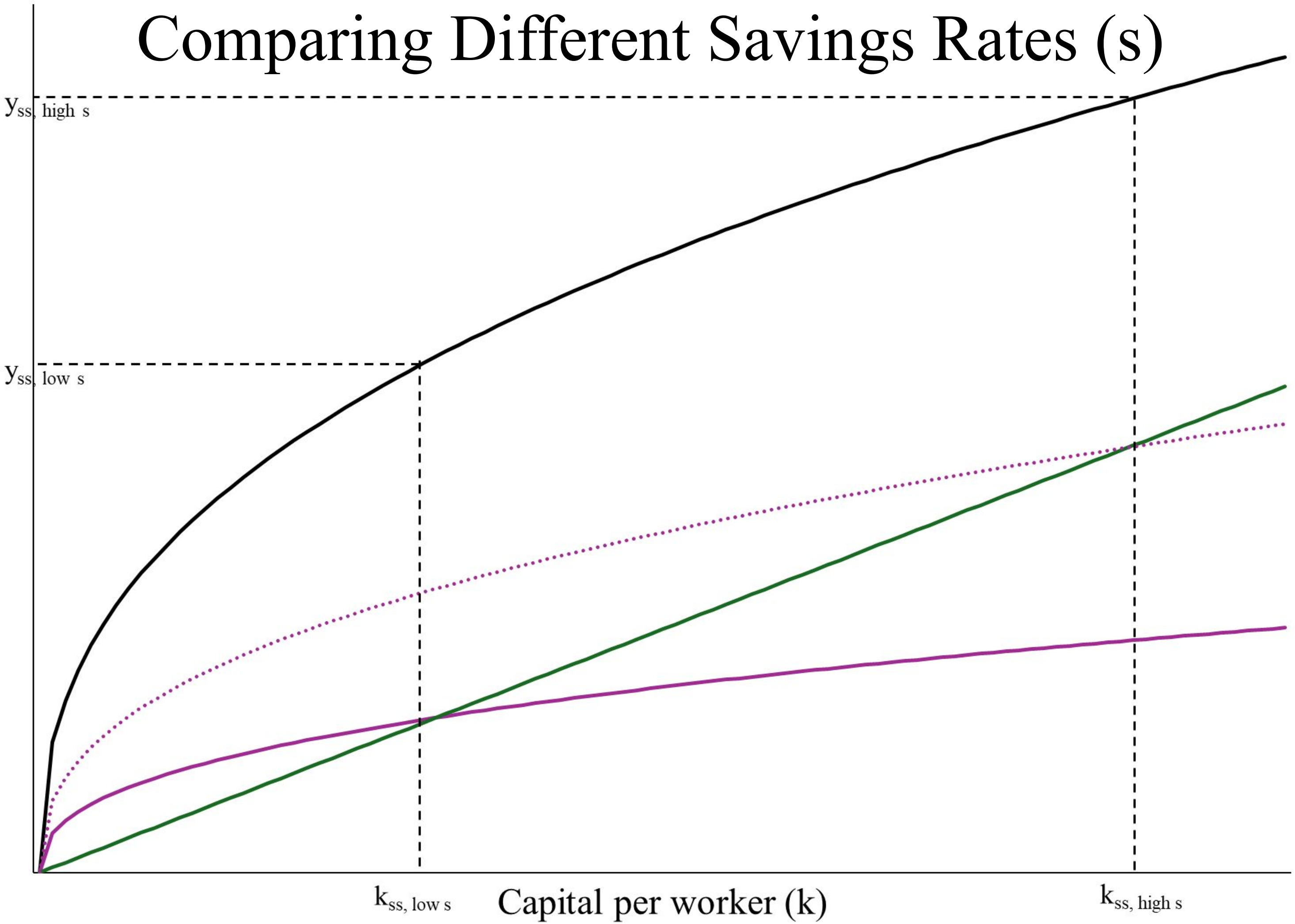
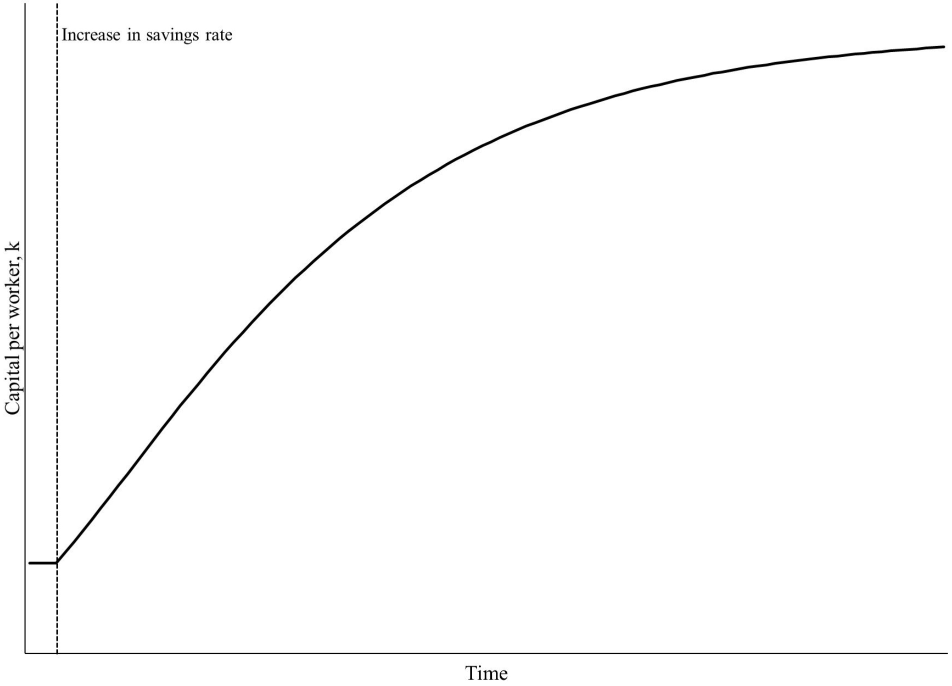


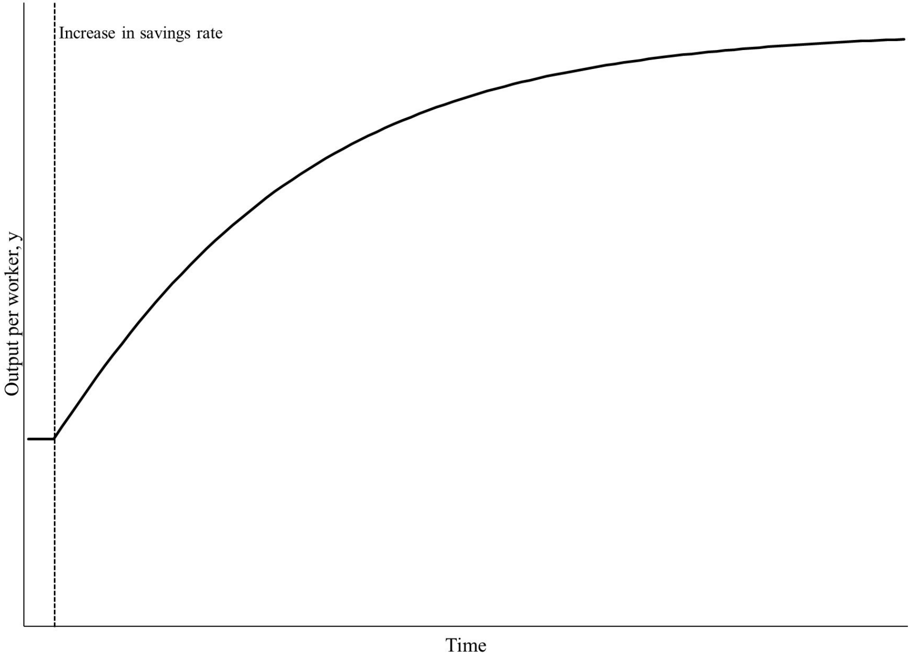
Comparing Different Savings Rates (s)



Transition Dynamics

- This analysis can partly explain the explosive growth in China in recent decades
- China had a very high savings rate, with low consumption and rapid capital accumulation

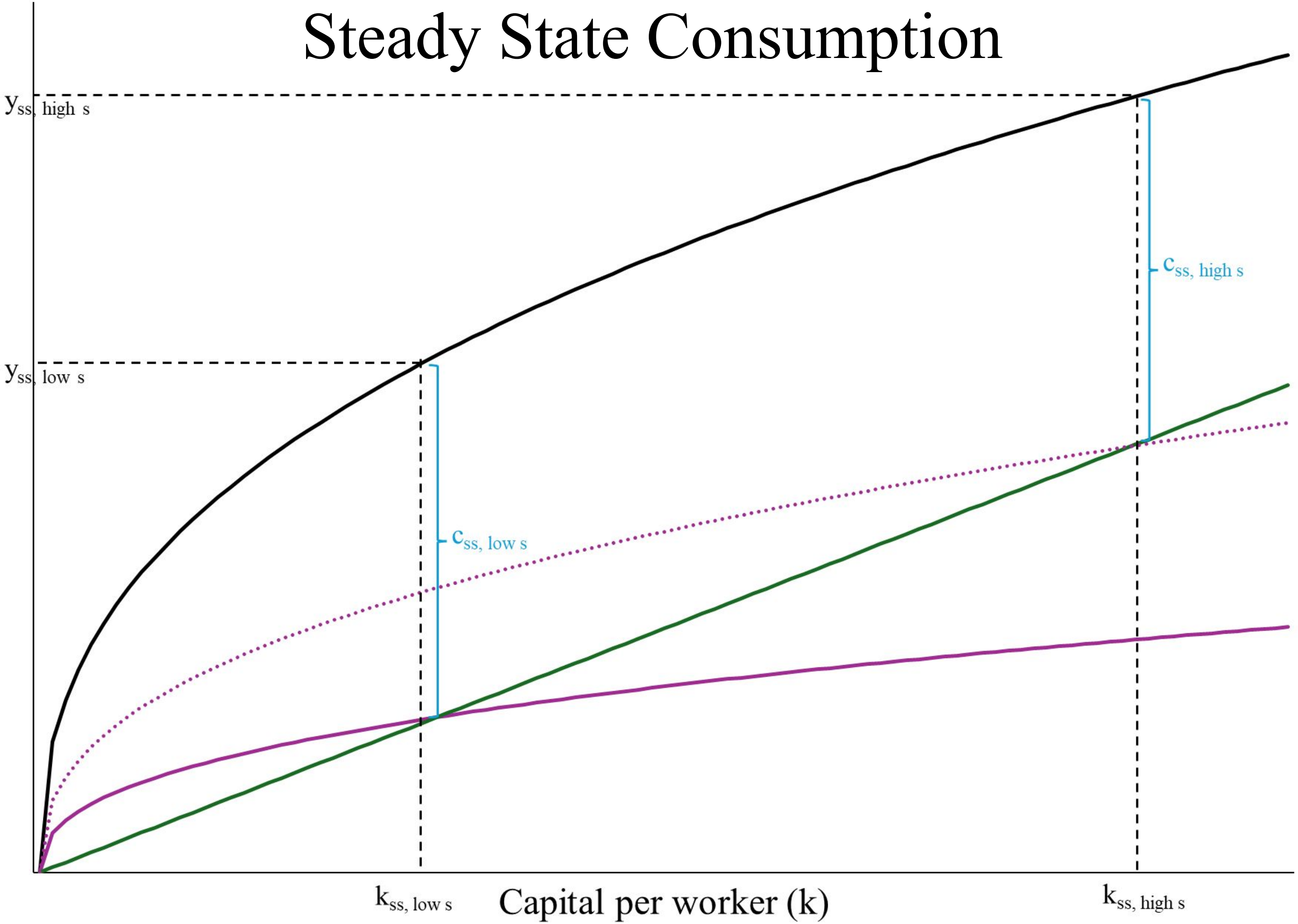




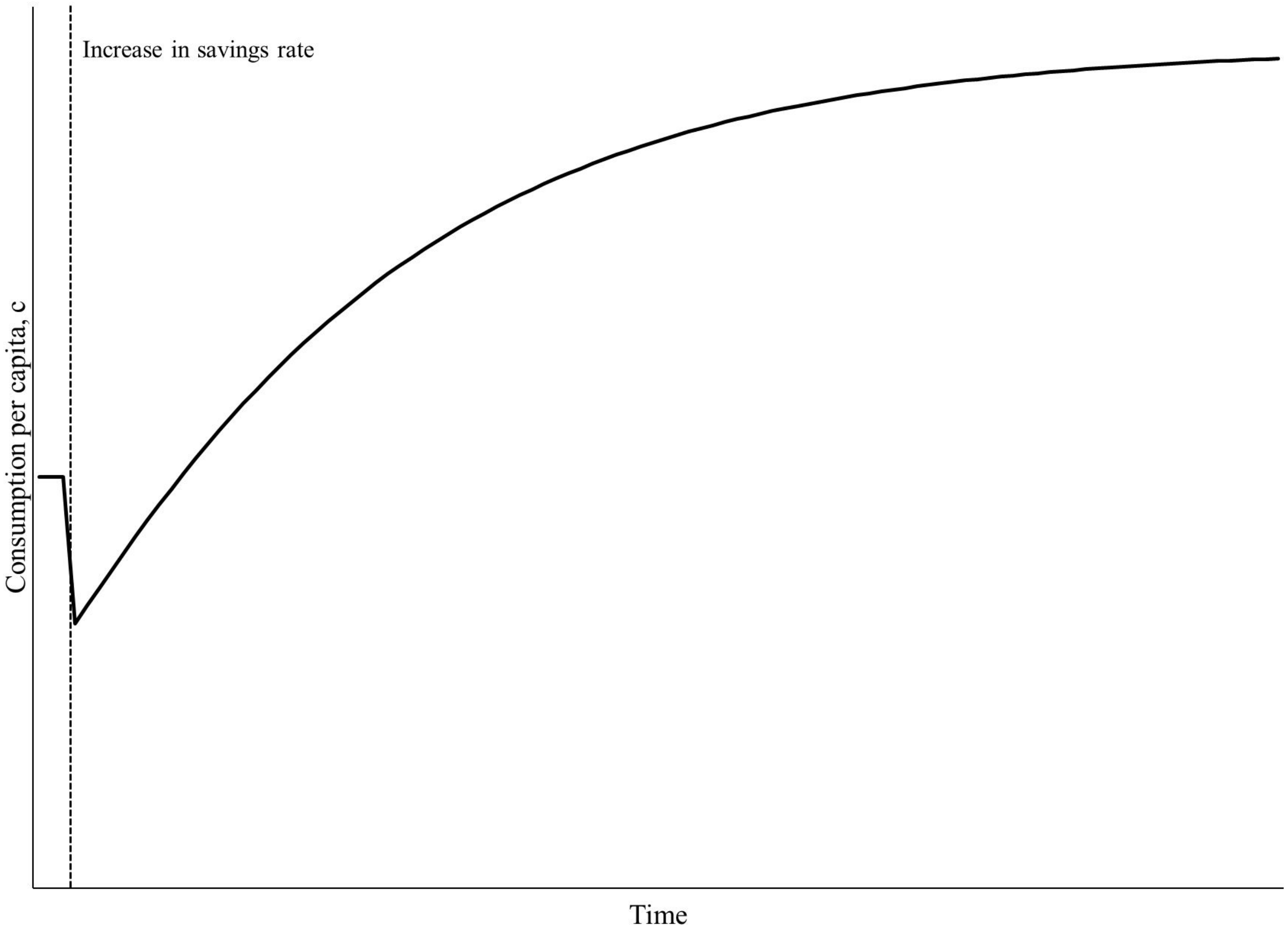
The Limits of Capital Accumulation

- Permanently increasing s will lead to:
  - Temporarily higher growth in k and y
  - Permanently higher levels of k and y (i.e. in steady state)
  - <u>No</u> change in long-run growth of k and y
    - There is still no growth in steady state
- Saving (investing) more eventually runs into diminishing marginal product
  - You have to save (invest) more and more to maintain the capital stock
  - Your capital becomes less and less productive
  - As a result, growth based on saving (investing) peters out
- What about the impact on consumption?

Steady State Consumption



Transition Dynamics of Consumption



The Golden Rule

- In steady state, savings (investment) is equal to depreciation, meaning:

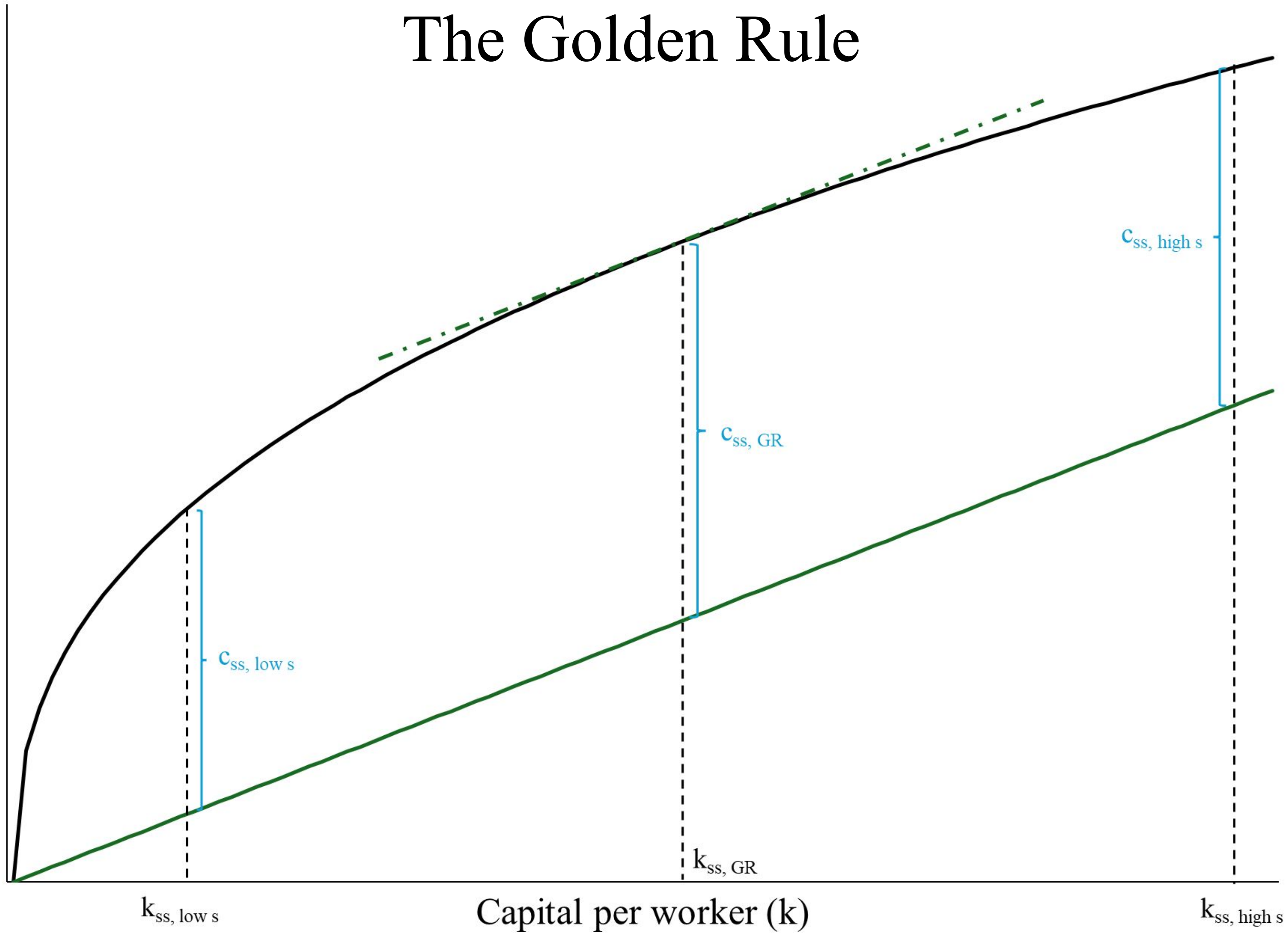
$$c_{ss} = f(k_{ss}) - \boldsymbol{\delta} * k_{ss}$$

- To maximize c_{ss}, take derivative wrt k_{ss} and set equal to 0:

$$f'(k_{GR}) - \boldsymbol{\delta} = 0 \qquad \text{OR} \qquad MP_k(k_{GR}) = \boldsymbol{\delta}$$

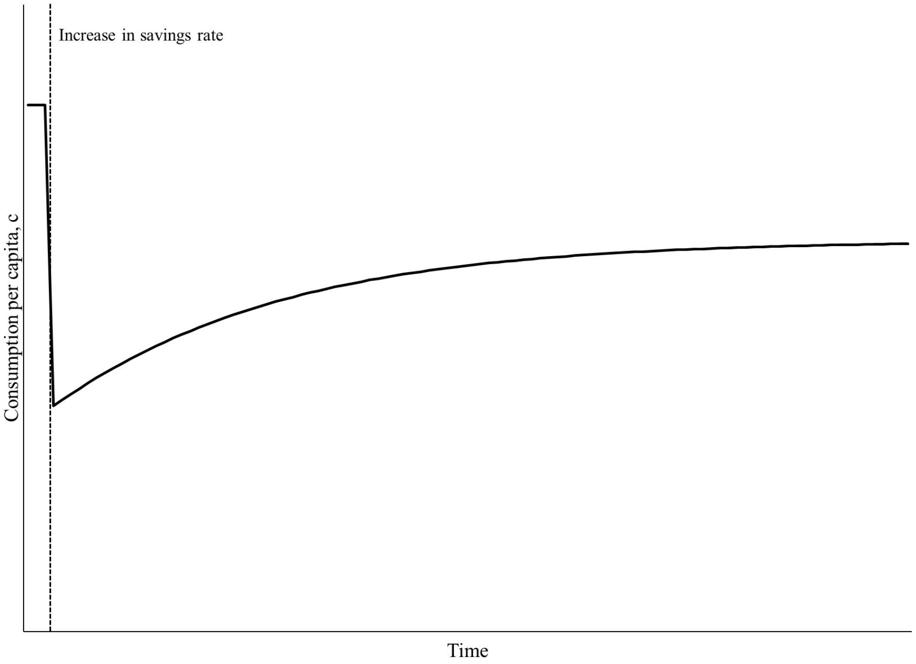
- Consumption is maximized when the marginal unit of capital can just barely sustain itself
  - i.e. it produces enough to recoup its own depreciation
  - If $k_{ss} < k_{GR}$, then $MP_k(k) > \boldsymbol{\delta}$
    - Marginal capital can sustain itself <u>and</u> generate some surplus: $k_{ss}\uparrow \longrightarrow c_{ss}\uparrow$
  - If $k_{ss} > k_{GR}$, then $MP_k(k) < \boldsymbol{\delta}$
    - Maintaining marginal capital requires output from other capital: $k_{ss}\downarrow \longrightarrow c_{ss}\uparrow$
  - Only at k_{GR} is there not a recipe to increase c_{ss}

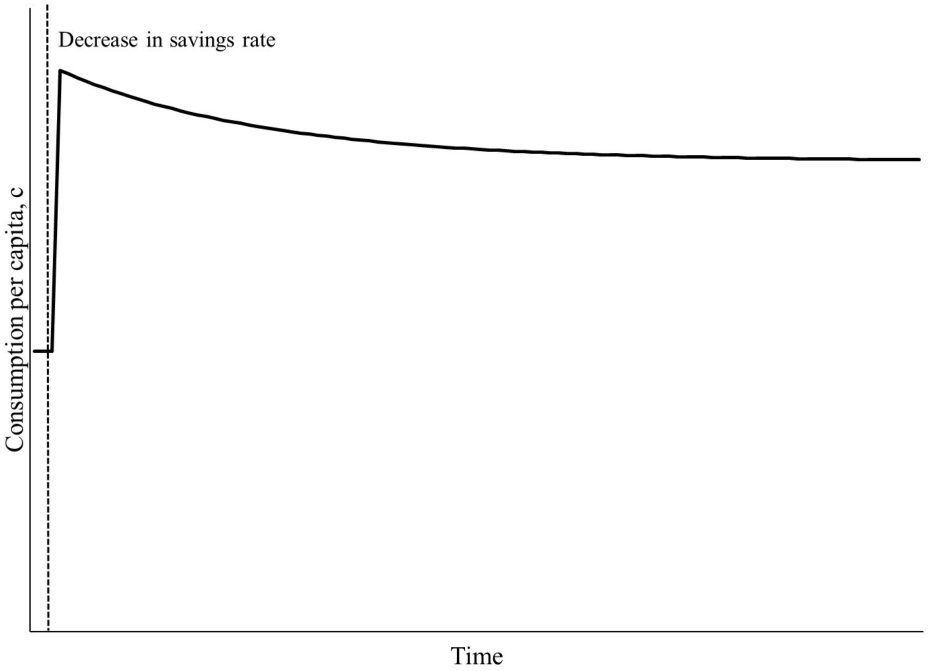
The Golden Rule



Dynamic Inefficiency

Starting from a very high savings rate:



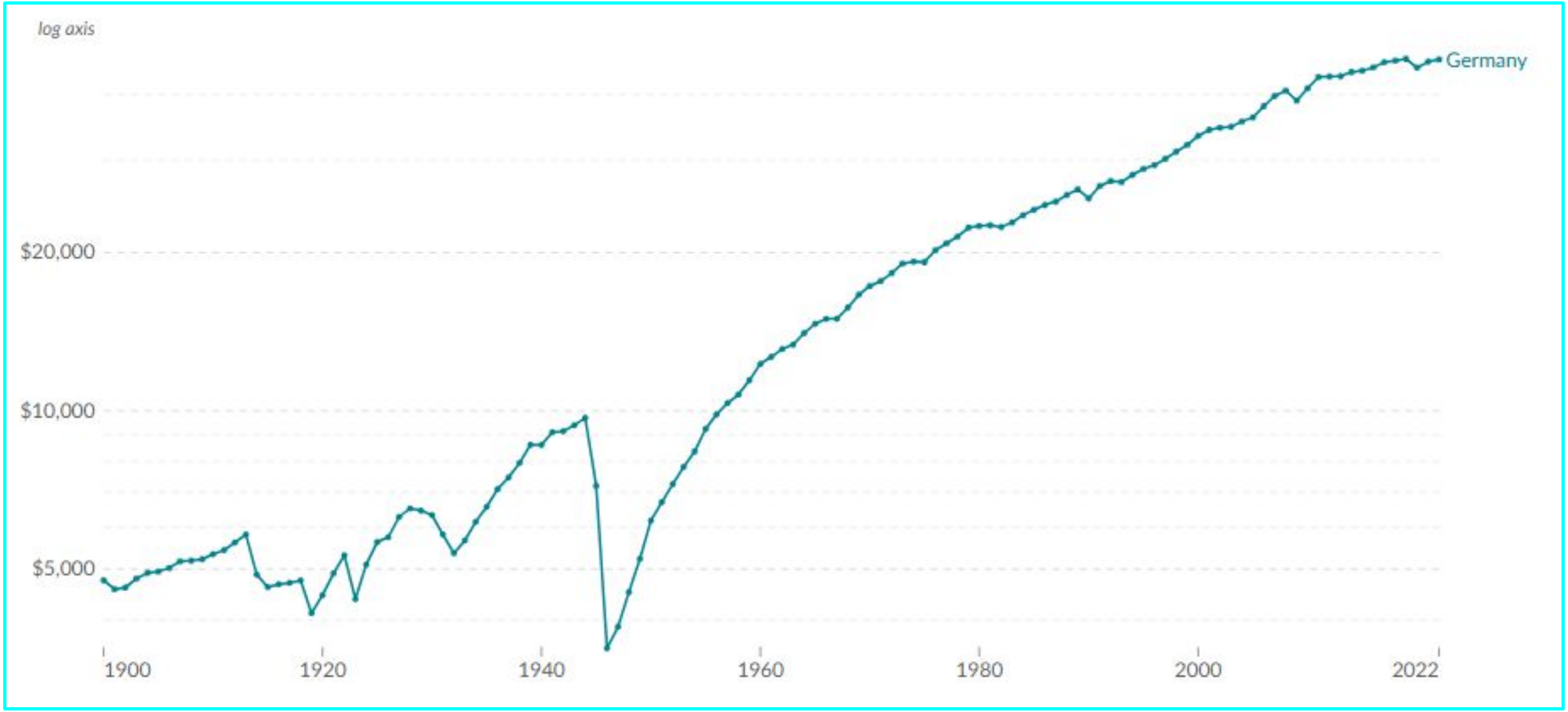


If you are ever in a position where saving less leads to more consumption <u>in all periods</u>, that means you have too much capital! That is called being "dynamically inefficient" – no matter your time preferences, there is no justification for so much capital. This happens because the marginal units just are not very productive.

Exercises

1. Suppose a war destroyed much of your capital stock, as in Europe (especially Germany) and Japan in WW2.
    - Assuming F and s stay constant, what would be the impact on future capital, output, and consumption?
2. Suppose there was a steep decline in population, as in Europe following The Plague.
    - Assuming F and s stay constant, what would be the impact on future capital, output, and consumption?

GDP in Germany



Sources:
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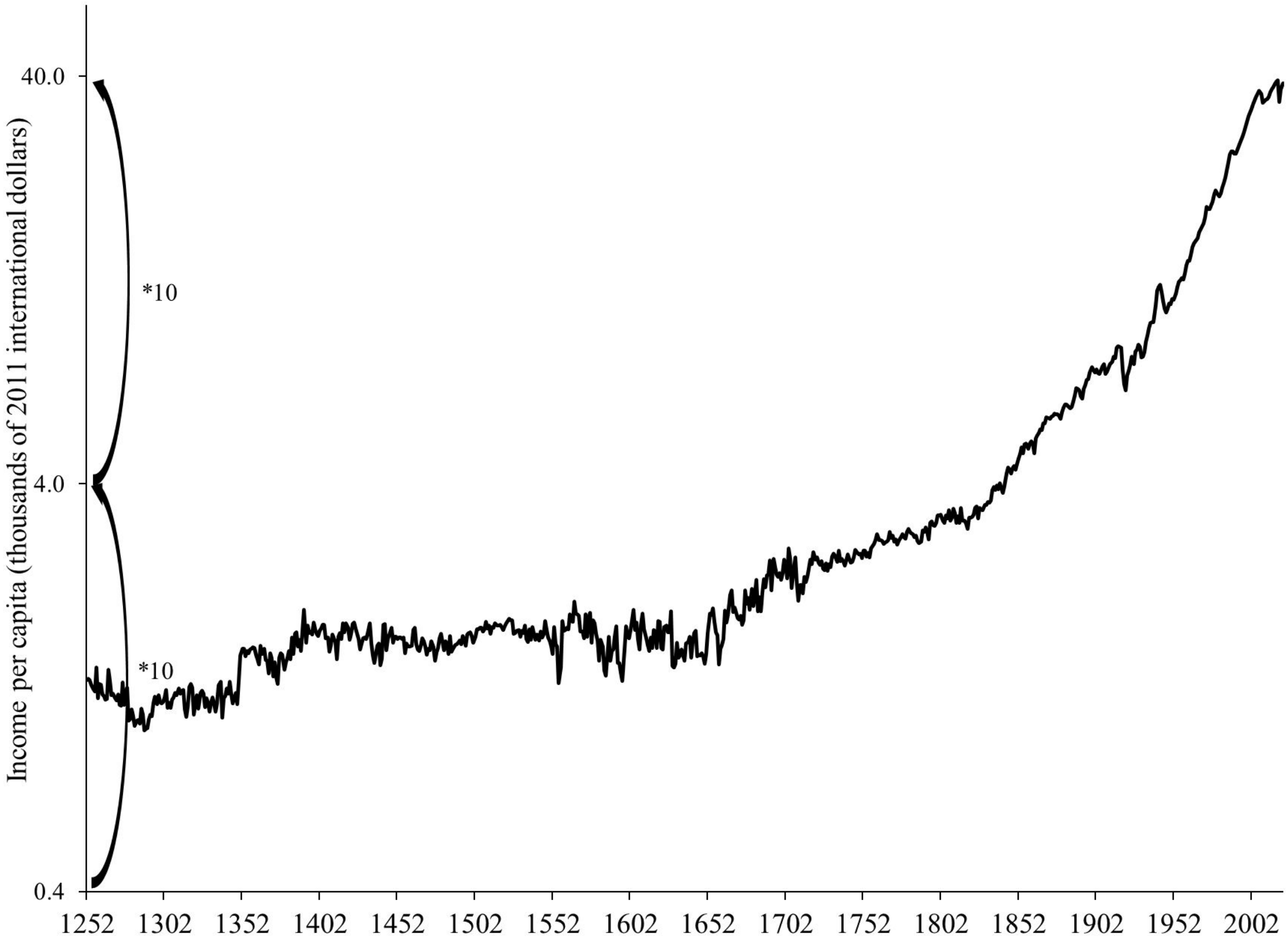
GDP in the UK



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Allowing Population Growth

- Now allow L to grow at some rate $g_L > 0$
- There will now be steady state growth in K
  - If K were constant, growing L would still increase Y and therefore I and therefore K…a contradiction!
- But what will happen to the per-capita values (k, y, and c)?
- Useful to recall the "Zipper Rule:"

$$g_{X\cdot Y} = \dot{ln\left(X \cdot Y\right)} = \dot{ln(X)} + \dot{ln(Y)} = g_X + g_Y$$

$$g_{X/Y} = \dot{ln\left(X/Y\right)} = \dot{ln(X)} - \dot{ln(Y)} = g_X - g_Y$$

- So capital per worker, k, now evolves as:

$$\dot{k}/k \equiv g_k = g_{K/L} = g_K - g_L \Rightarrow \dot{k} = (g_K - g_L) \cdot k$$

Break-even Investment with Population Growth

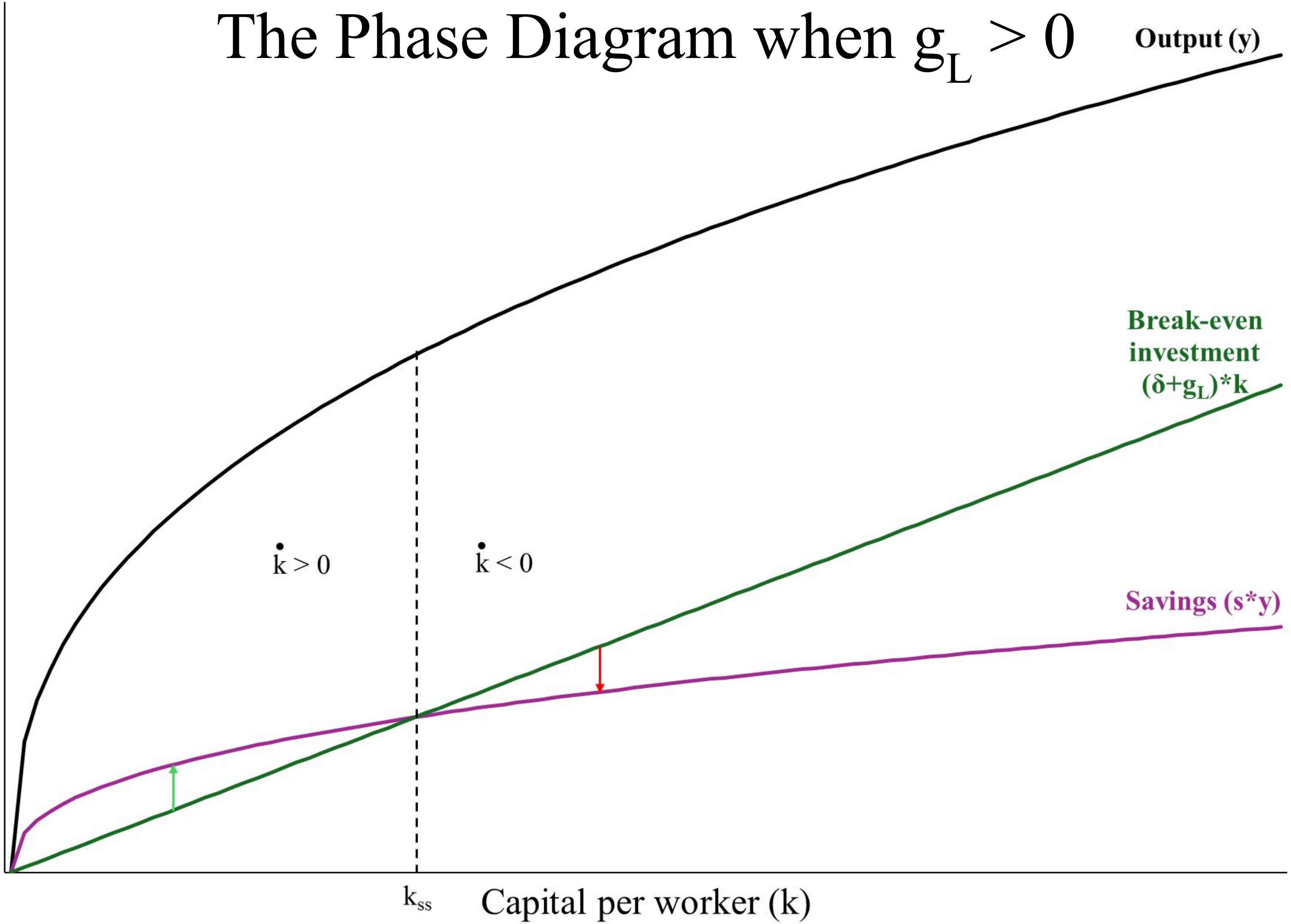
$$\begin{aligned}\dot{k} &= (g_K - g_L) \cdot k \\ &= \frac{\dot{K}}{K} \cdot \frac{K}{L} - g_L \cdot k \\ &= \frac{s \cdot F(K, L) - \delta \cdot K}{L} - g_L \cdot k \\ &= s \cdot f(k) - \delta \cdot k - g_L \cdot k \\ &= s \cdot f(k) - (\delta + g_L) \cdot k\end{aligned}$$

- Therefore, break-even investment occurs when:

$$s \cdot f(k_t) = (\delta + g_L) \cdot k_t$$

- Intuitively, keeping per-capita capital constant requires:
    1. Replacing the depreciated capital
    2. Equipping new workers with that level of capital

The Phase Diagram when $g_L > 0$



Balanced Growth Path when $g_L > 0$

- On this "balanced growth path" (BGP), we have $g_K - g_L = 0$, so $g_K = g_L$.
- But per-capita capital – and therefore per-capita output and consumption – are still constant on this BGP

Capital per Effective Worker

- Now suppose E_t grows at rate g_E.
- BGP will no longer have constant k
  - If k were constant, "effectiveness" improvement would still increase y and therefore i and therefore k…a contradiction!
- Now define something called capital per effective worker:

$$\hat{k}_t \equiv \frac{K_t}{E_t \cdot L_t}$$

- Zipper rule tells us that

$$\dot{\hat{k}}/\hat{k} \equiv g_{\hat{k}} = g_K - g_L - g_E \Rightarrow \dot{\hat{k}} = (g_K - g_L - g_E) \cdot \hat{k}$$

- Defining output per effective worker as $\hat{y}_t \equiv \frac{Y_t}{E_t \cdot L_t}$, we get:

$$\begin{aligned} \hat{y} &= \frac{F(K, E \cdot L)}{E \cdot L} \\ &= F\left(\frac{K}{E \cdot L}, 1\right) \\ &= f(\hat{k}) \end{aligned}$$

Break-even Investment with Effectiveness Growth

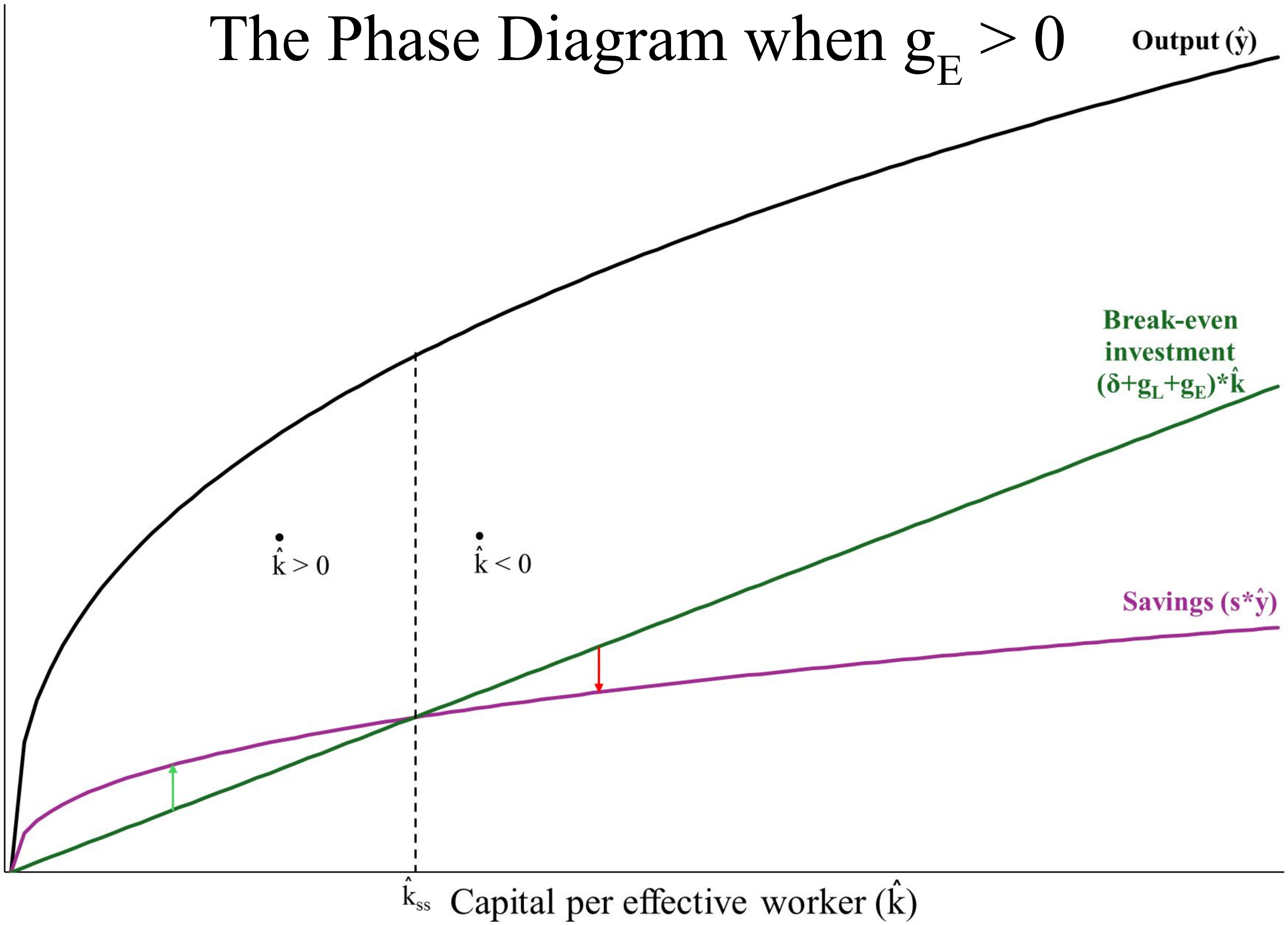
$$
\begin{aligned}
\dot{\hat{k}} &= (g_K - g_L - g_E) \cdot \hat{k} \\
&= \frac{\dot{K}}{K} \cdot \frac{K}{E \cdot L} - (g_L + g_E) \cdot \hat{k} \\
&= \frac{s \cdot F(K, E \cdot L) - \delta \cdot K}{E \cdot L} - (g_L + g_E) \cdot \hat{k} \\
&= s \cdot f(\hat{k}) - \delta \cdot \hat{k} - (g_L + g_E) \cdot \hat{k} \\
&= s \cdot f(\hat{k}) - (\delta + g_L + g_E) \cdot \hat{k}
\end{aligned}
$$

- Therefore, break-even investment occurs when:

$$s \cdot f(\hat{k}_t) = (\delta + g_L + g_E) \cdot \hat{k}_t$$

- Intuitively, keeping capital per effective worker constant requires:
    1. Replacing the depreciated capital
    2. Equipping new workers with that level of capital
    3. Upgrading existing capital to keep up with improving workers

The Phase Diagram when $g_E > 0$



Balanced Growth Path when $g_E > 0$

- On this "balanced growth path" (BGP), we have $g_K - g_L - g_E = 0$, so $g_K = g_L + g_E$.
- So now per-capita capital grows!
  - $g_{K/L} = g_K - g_L = g_E$
- So does per-capita output:
  - $g_{Y/L} = g_{Y/(EL)} + g_E = 0 + g_E = g_E$
- So does per-capita consumption:
  - $g_{C/L} = g_{(1-s)Y/L} = g_{(1-s)} + g_{Y/L} = 0 + g_E = g_E$

Exercise

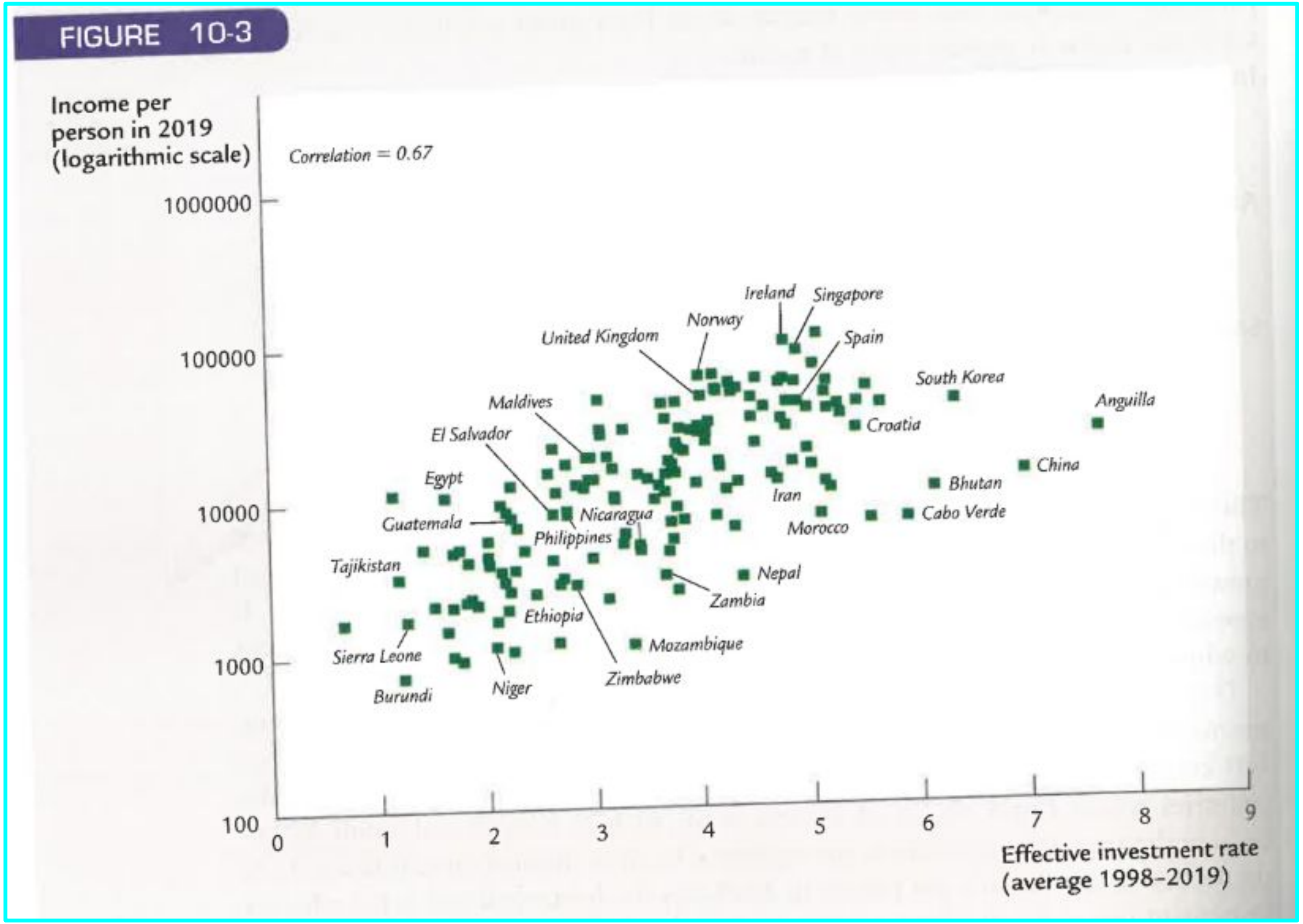
- Assume you start on a balanced growth path, and then g_E suddenly increases. Plot the time paths of:
  1. Capital per effective worker
  2. Capital per worker
  3. Output per worker

Takeaways from the Theory

- More investment leads to more capital and output
  - Whether it leads to more consumption in the long-run depends on how much capital you start with
    - Recall the analysis of the Golden Rule…
- This is a long-run effect on the <u>levels</u> of capital and output – long-run <u>growth</u> is not spurred by investment
  - As capital stock grows, the process slows down and eventually plateaus
- Sustained <u>growth</u> comes from improving the “special sauce” we’ve been calling “effectiveness”

Back to Data

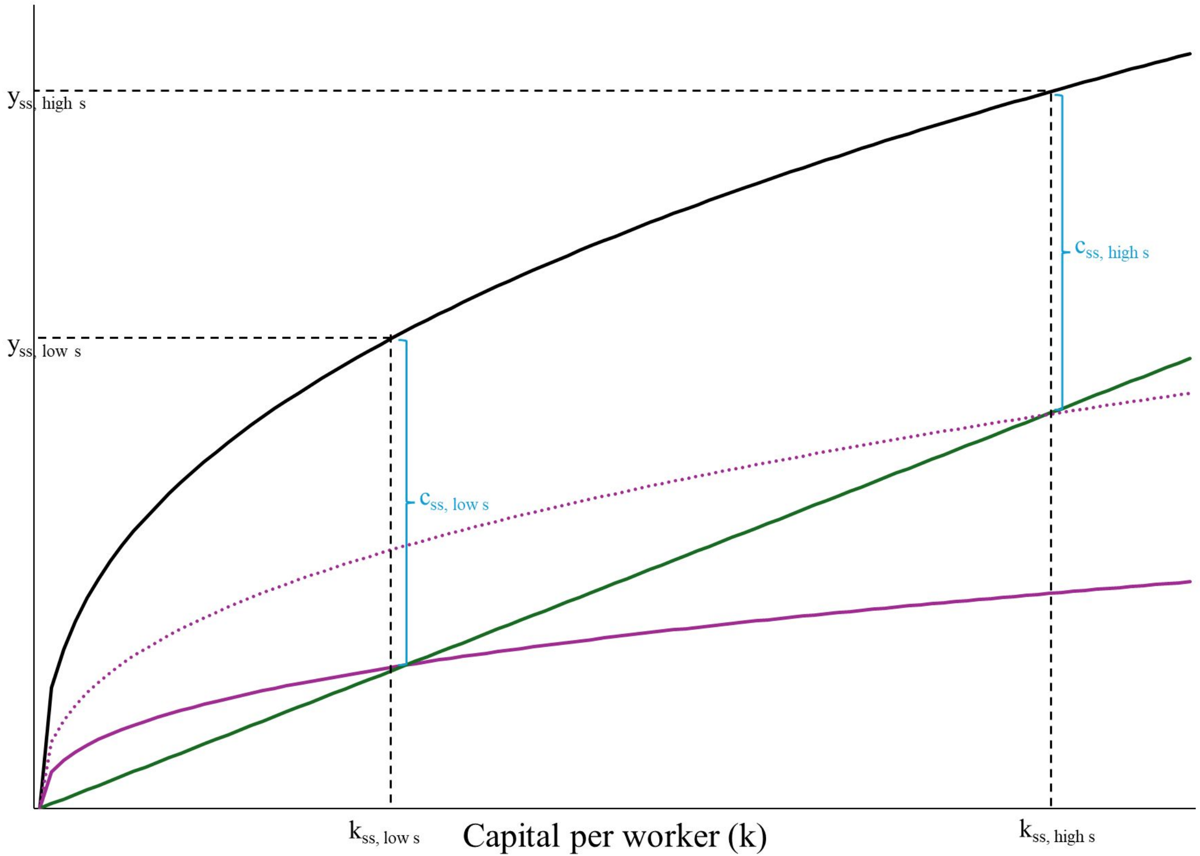
Higher savings is associated with higher income levels



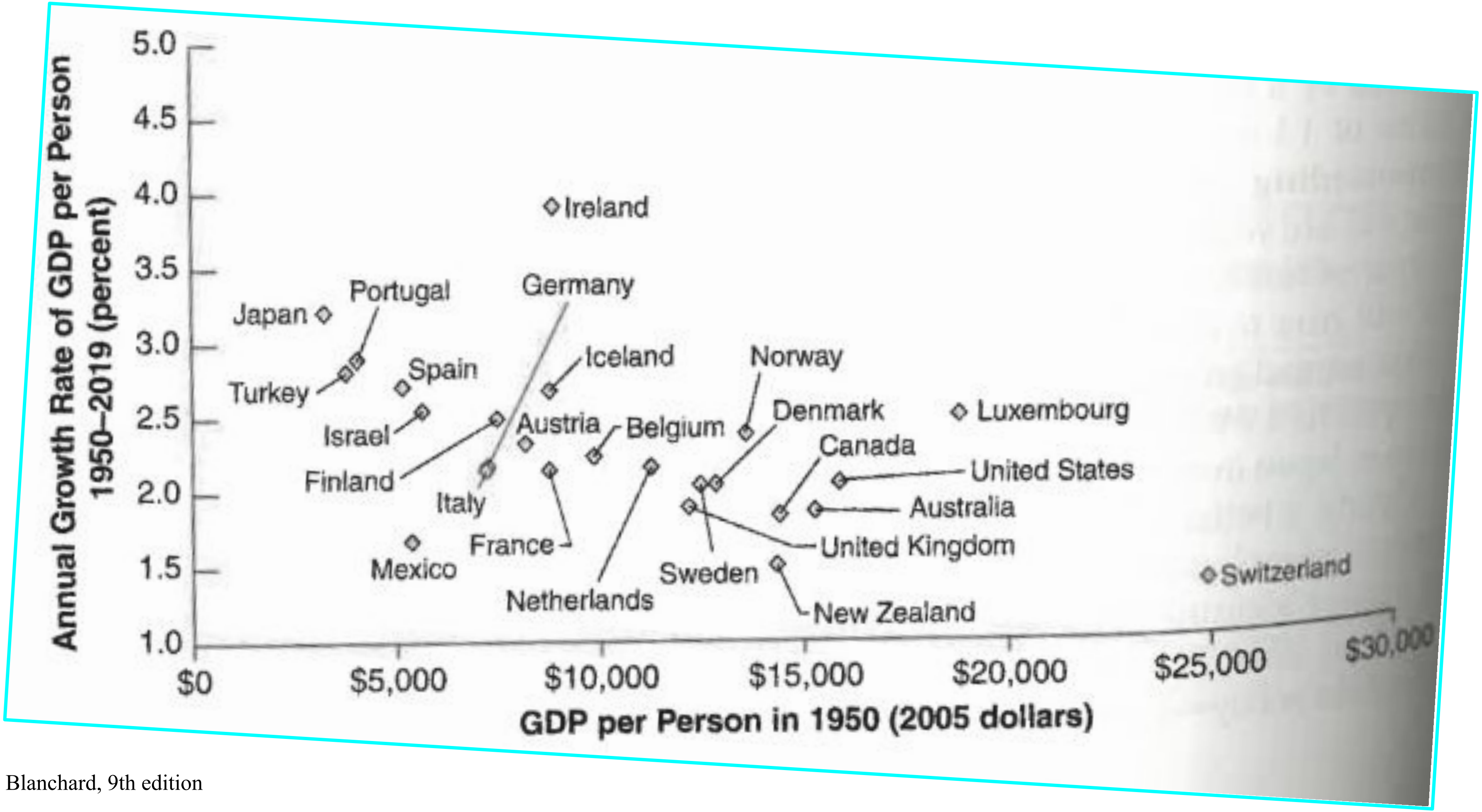
Source: Your textbook

Recall this:

1. More savings leads to more capital accumulation and more output



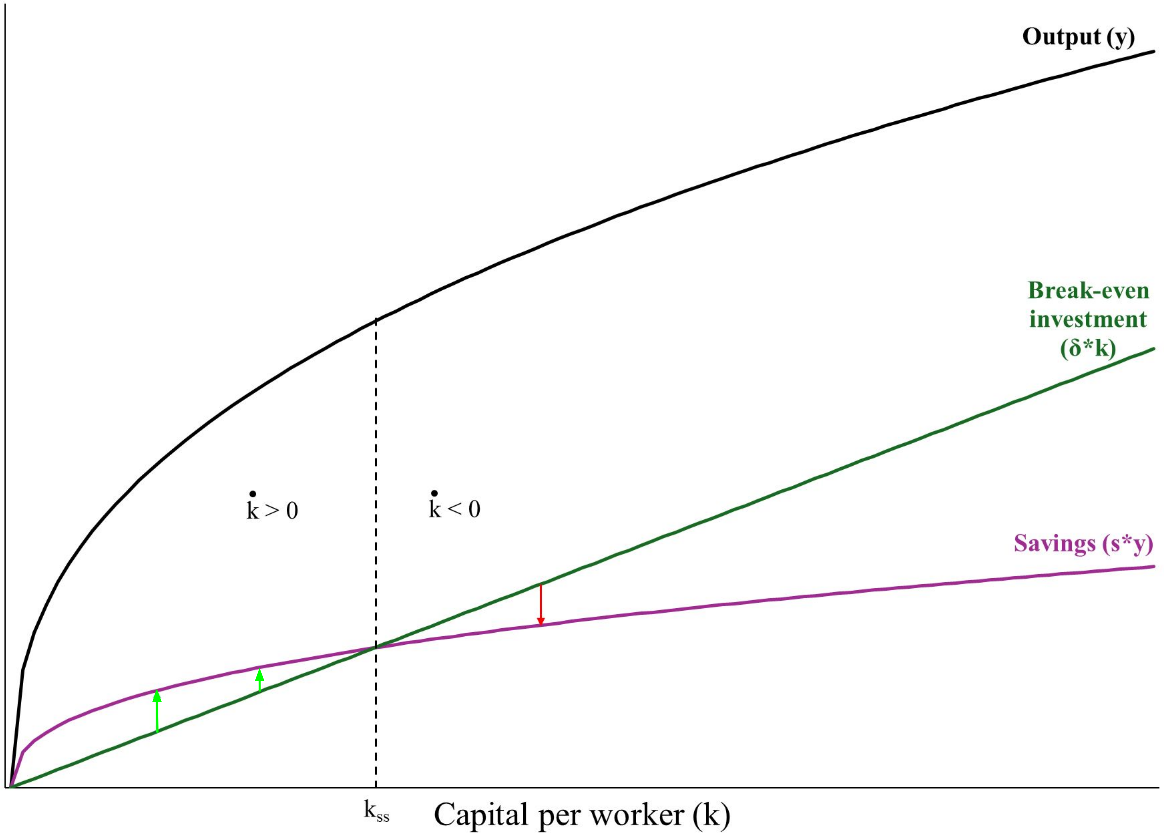
But within OECD, poorer countries are catching up to richer ones



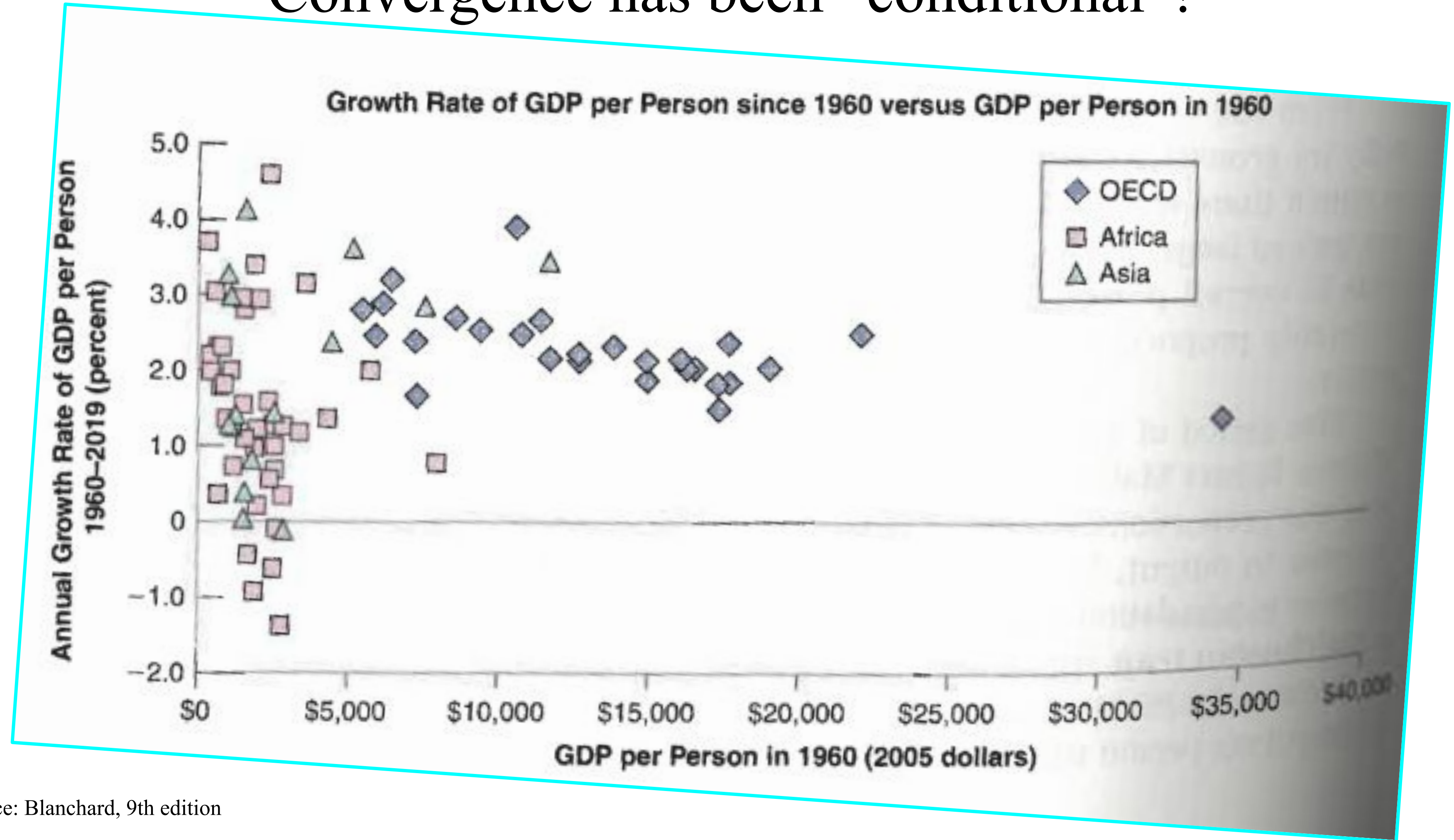
Source: Blanchard, 9th edition

Recall this:

1. More savings leads to more capital accumulation and more output
2. Growth slows down as a country approaches its steady state
   - More capital means more depreciation…
   - This is known as <u>convergence</u>

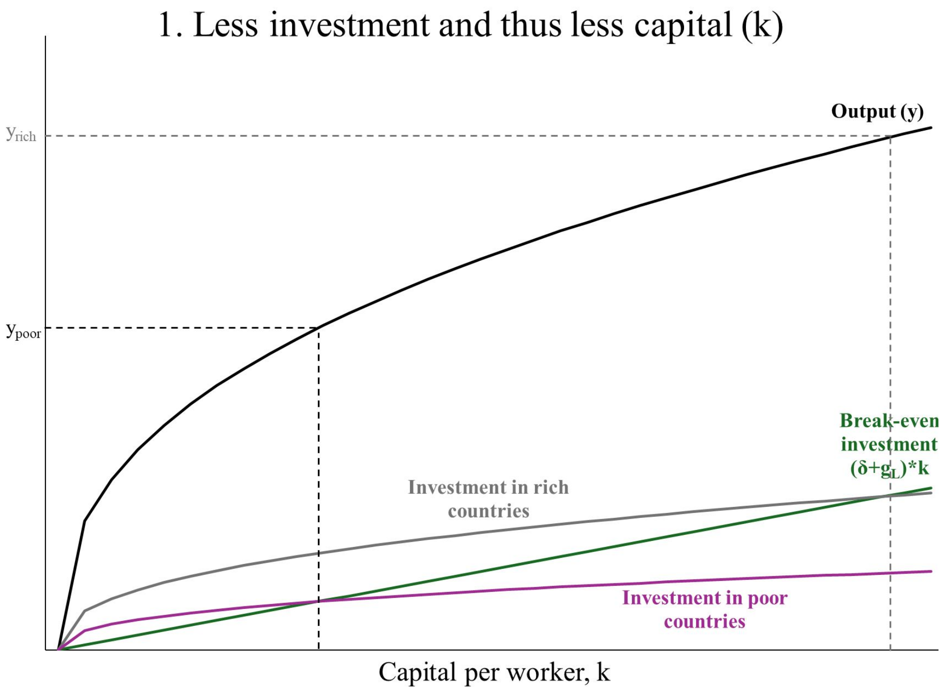


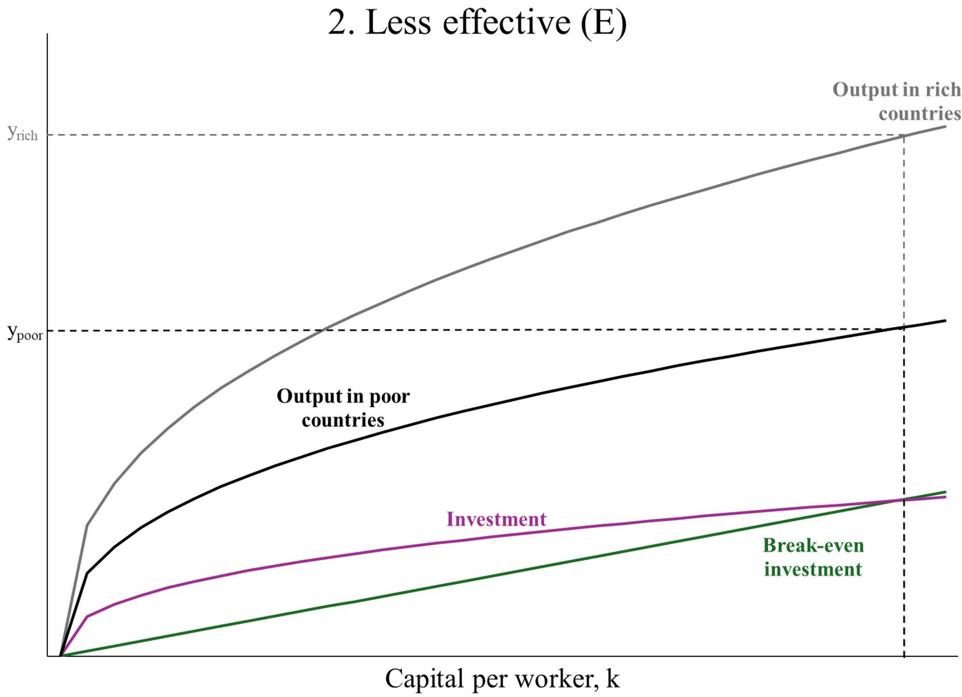
Convergence has been "conditional"!



Source: Blanchard, 9th edition

Solow Model suggests 2 main reasons for lack of convergence





Growth Accounting

Cobb-Douglas Production

- Most common production is Cobb-Douglas:

$$Y = K^{\alpha} \cdot (E \cdot L)^{1-\alpha}$$

- Define $A \equiv E^{1-\alpha}$, so:

$$Y = A \cdot K^{\alpha} \cdot L^{1-\alpha}$$

- $\boldsymbol{\alpha}$ is known as "capital's share," with (1-$\boldsymbol{\alpha}$) being "labor's share"
  - In a competitive market, the wage paid to labor will be its marginal product:

$$w = MP_L = \frac{\partial Y}{\partial L} = (1-\alpha) \cdot A \cdot \left(\frac{K}{L}\right)^{\alpha}$$

  - So total payments to labor are:

$$w \cdot L = (1-\alpha) \cdot A \cdot K^{\alpha} \cdot L^{1-\alpha}$$

  - So labor's share of total output is:

$$\frac{w \cdot L}{Y} = \frac{(1-\alpha) \cdot A \cdot K^{\alpha} \cdot L^{1-\alpha}}{A \cdot K^{\alpha} \cdot L^{1-\alpha}} = 1-\alpha$$

Solow Residual

- Per-capita output with C-D production:

$$\frac{Y}{L} = A \cdot \left(\frac{K}{L}\right)^{\alpha}$$

- Zipper rule gives:

$$g_{\frac{Y}{L}} = \alpha \cdot g_{\frac{K}{L}} + g_A$$

- In an "accounting" sense, growth in output per worker comes from two sources:
  1. Capital deepening: equipping workers with more capital
  2. Growth in the "Solow Residual"
     - i.e. everything that isn't capital!
       - (We were calling it "effectiveness" earlier.)
       - Oftentimes also called "Total Factor Productivity" (TFP)
       - The stuff we can't really explain

Growth Accounting

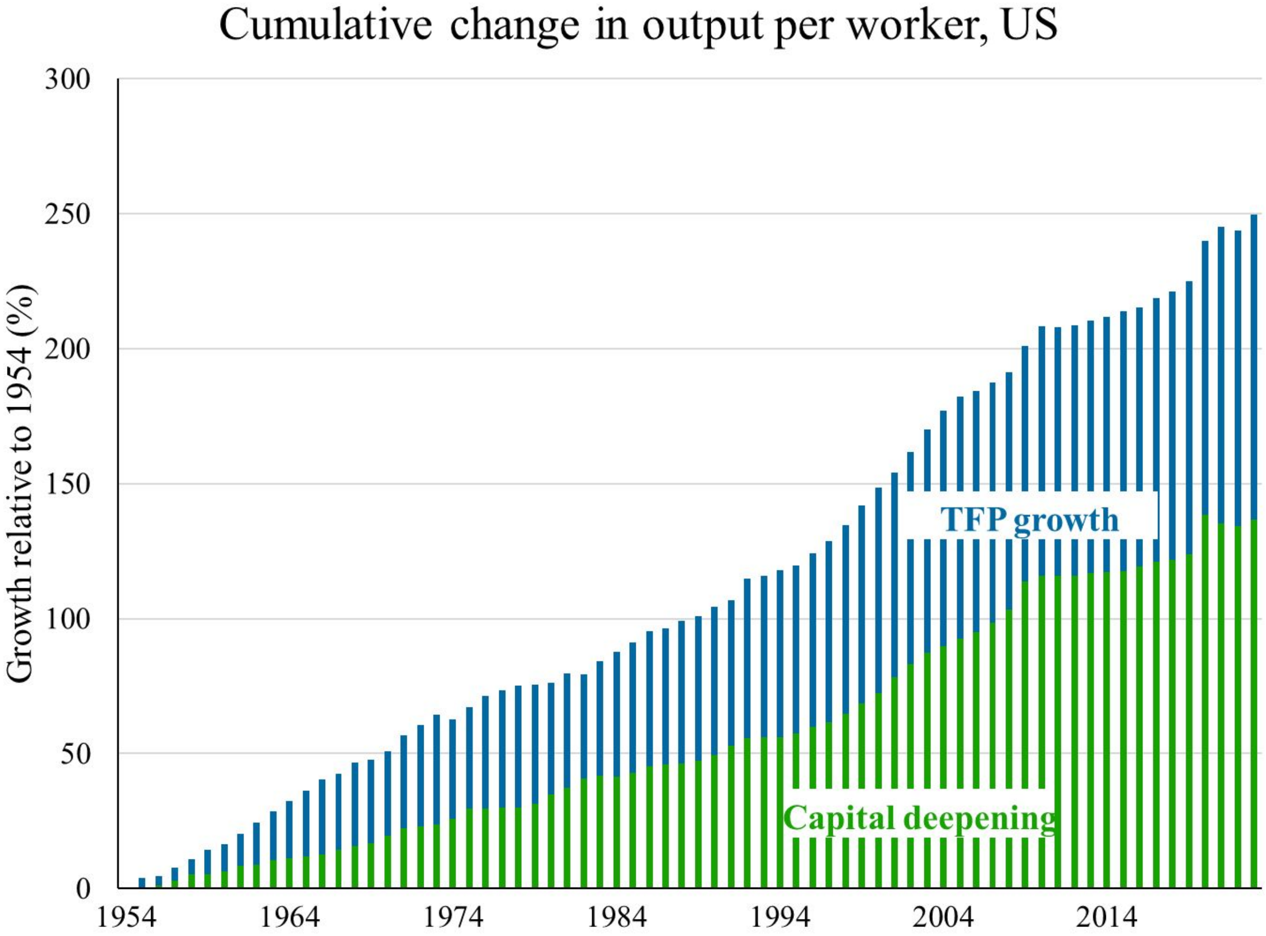
- Labor's share in the U.S. has held relatively steady at about ⅔, i.e. ***a*** = ⅓ . So:

$$g_{\frac{Y}{L}} = \frac{1}{3} \cdot g_{\frac{K}{L}} + g_A$$

- Can isolate the drivers of growth in an economy over time:
  - Choose a base year, e.g. 1954
  - For any year since then, we can compute the growth in output- and capital-per-worker since 1954
    - E.g. 1973: $g_{Y/L}$ = 64%, $g_{K/L}$ = 62%
  - Compute the unexplained growth (i.e. TFP growth) using the above formula
    - g_A = 64% - ⅓ *62% = 41%
  - Decompose income-per-capita growth between capital deepening and TFP
    - 64% = ⅓ *62% + 41%
      - ➢ Of the 64% growth during that period, 37% came from capital deepening and 63% from TFP

Growth in the US has come from both capital and technology

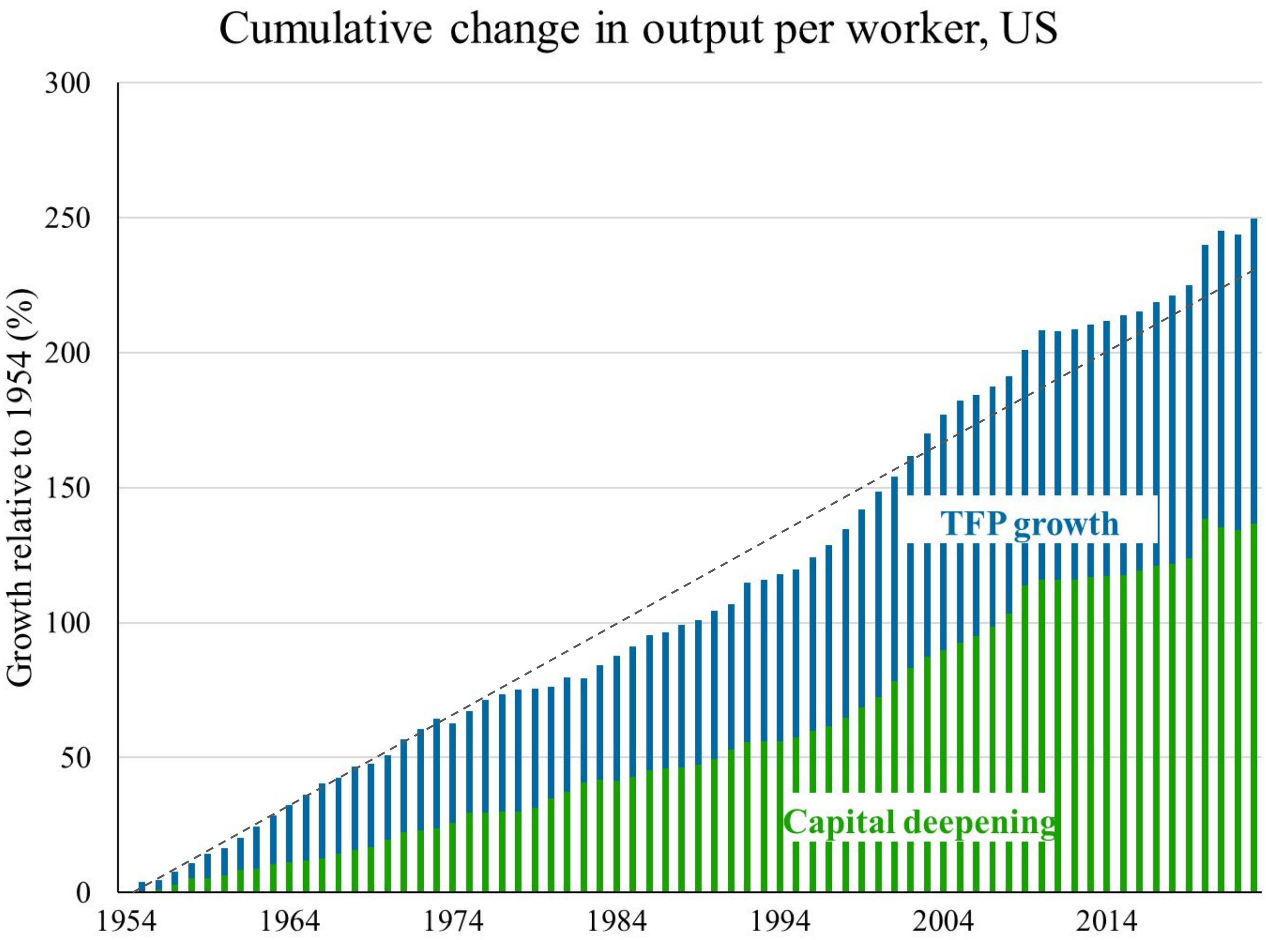
- TFP was the main driver for the first ~20 years
- Capital deepening has been more important since the mid-1970s
- But overall, the two seem roughly equal



Source: Penn World Tables

Growth Slowdown

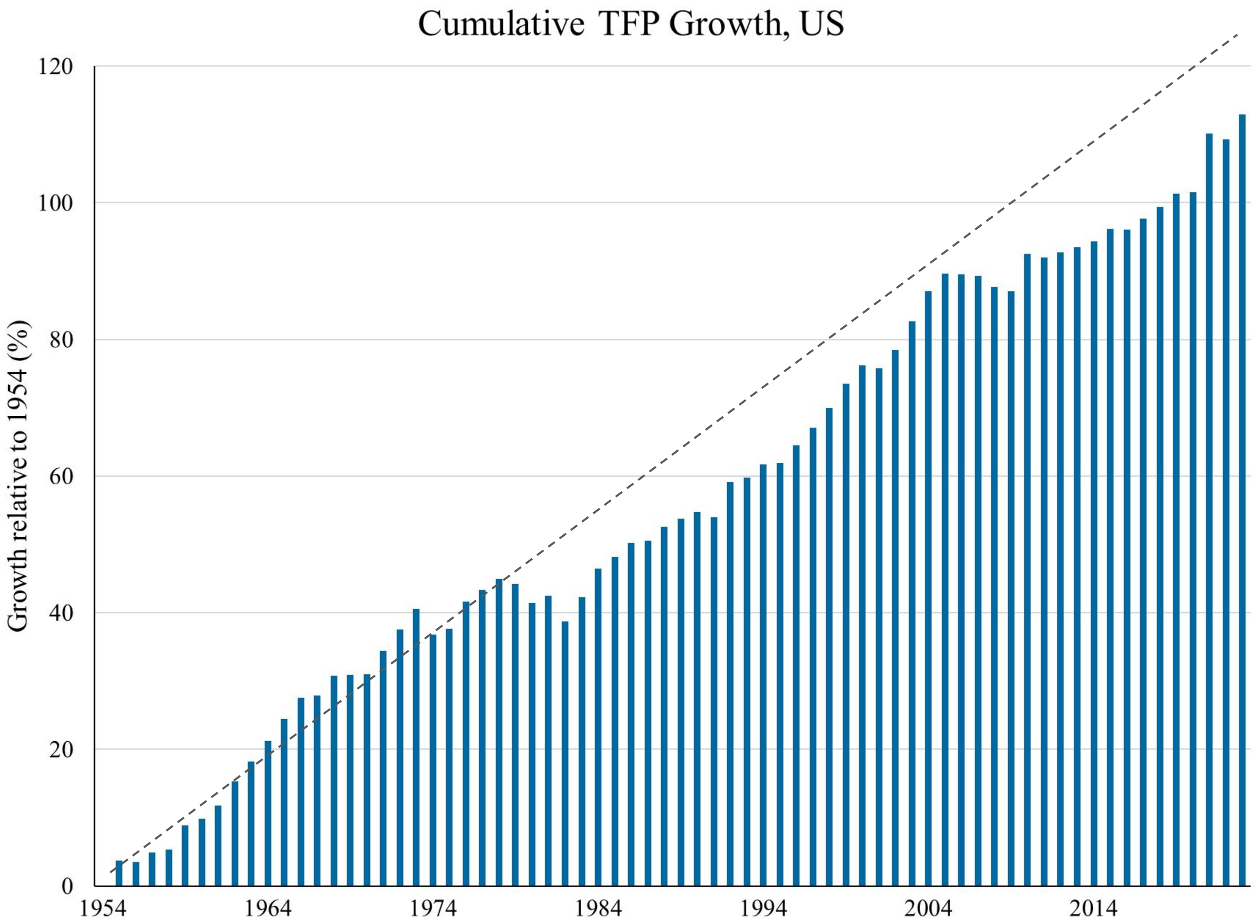
- Growth slowed in the mid-1970s



Source: Penn World Tables

Growth Slowdown

- Growth slowed in the mid-1970s
- This was attributable to the slowdown in TFP growth
- Besides late 1990s, TFP growth has never really recovered



Source: Penn World Tables

Accounting for Disparities Across Countries

Country	Per capita GDP (y)	$\bar{k}^{1/3}$	Implied TFP ($\bar{A}$)
United States	1.000	1.000	1.000
Switzerland	1.151	1.094	1.052
U.K.	0.714	0.885	0.807
Japan	0.734	0.976	0.752
Italy	0.680	0.930	0.731
Spain	0.640	0.901	0.710
China	0.279	0.651	0.429
Brazil	0.252	0.587	0.429
South Africa	0.214	0.559	0.383
India	0.117	0.433	0.270
Burundi	0.015	0.173	0.084

Source: Chad Jones MgtEcon 300 lecture slides from 2023, Session 2, slide 29

Accounting for Disparities Across Countries (2)

- Compare the five richest and five poorest economies:

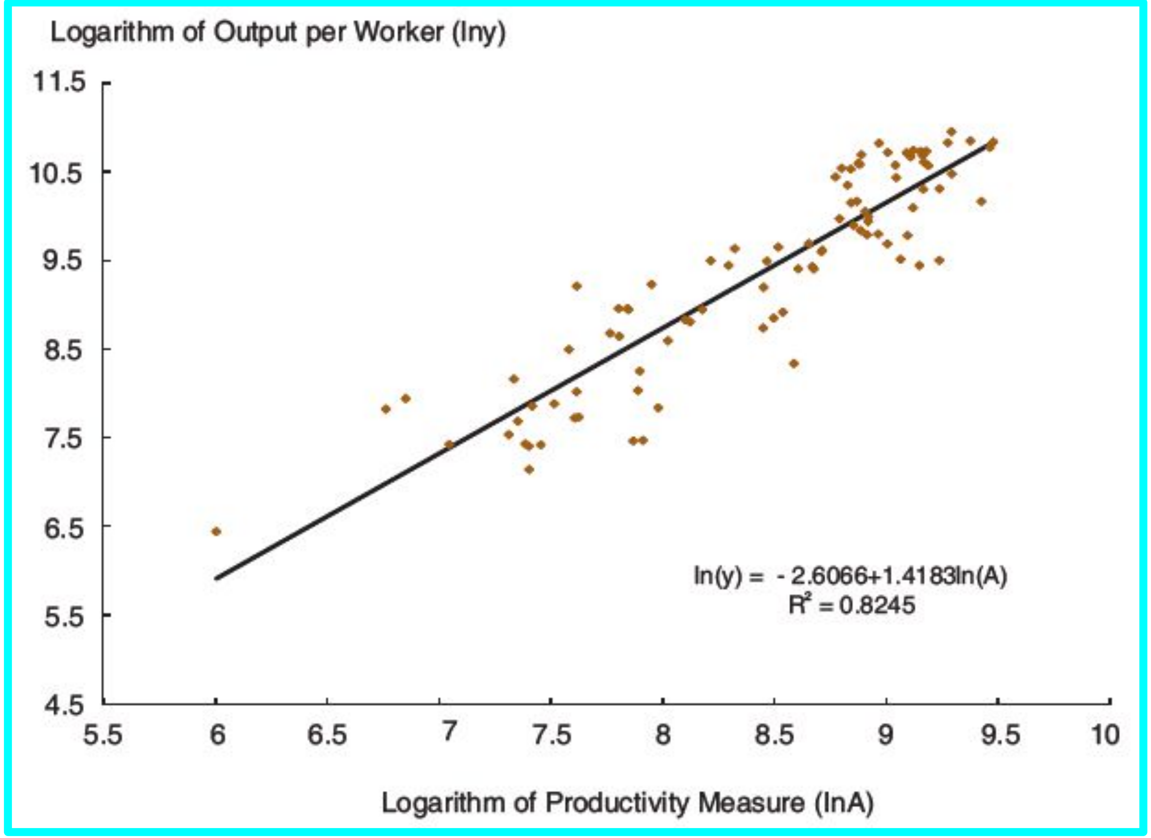
$$\underbrace{\frac{y_{rich}}{y_{poor}}}_{65} \cong \underbrace{\frac{\bar{A}_{rich}}{\bar{A}_{poor}}}_{13} \cdot \underbrace{\left(\frac{k_{rich}}{k_{poor}}\right)^{1/3}}_{5}$$

Source: Chad Jones MgtEcon 300 lecture slides from 2023, Session 2, slide 32

Accounting for Disparities Across Countries (3)

$$\frac{Y}{L} = A \cdot \left(\frac{K}{L}\right)^{\alpha} \longrightarrow \ln\left(\frac{Y}{L}\right) = \alpha \cdot \ln\left(\frac{K}{L}\right) + \ln(A)$$

- TFP explains over 80% of why some countries are rich while others are poor!
- It’s (mostly) not about having more capital, it’s about using it better.



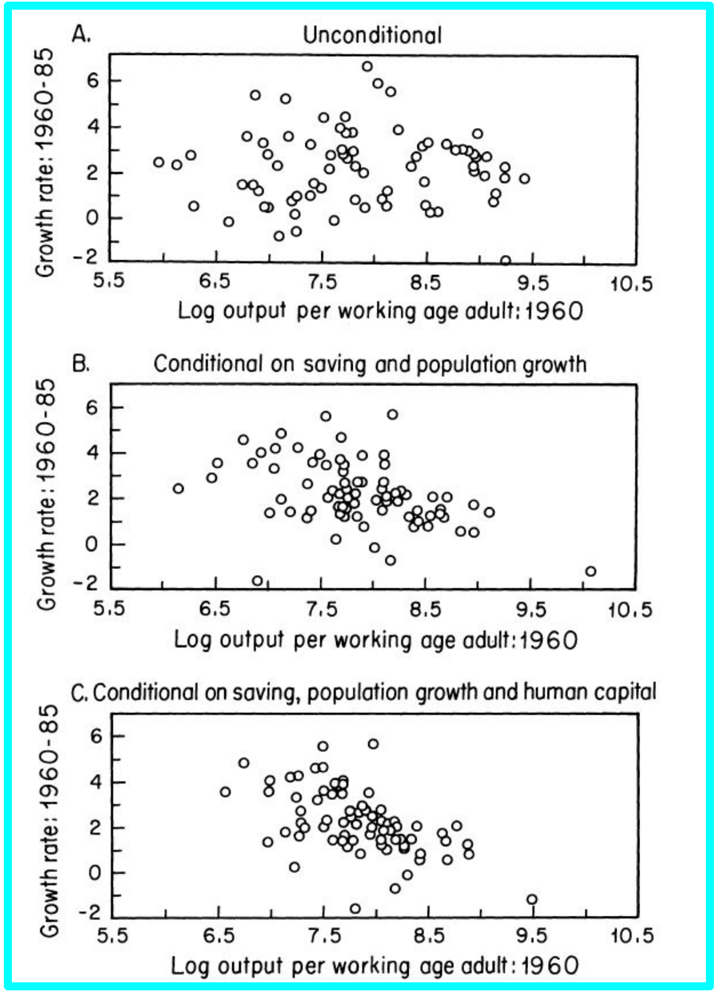
Source: Khan (2009)

What is the Solow Residual?

- This residual is the dominant force in growth and development; so what is it?!
- Often described as “technology” but beware an important subtlety:
  - A fancier computer with better specs is actually captured as <u>more capital</u>
  - Giving a computer access to the internet or AI will improve its effectiveness – that would appear in the Solow Residual
- Education of workers
- Managerial effectiveness
- Optimized production processes (often via “learning-by-doing”)
- Data
- Infrastructure
- Resource allocation (e.g. are most effective firms and workers able to expand?)
  - Corruption is part of this: are you successful because you’re productive, or because you’re connected?
- Mismeasurement
  - This is a boring and technical point, but it is important: it truly is a “residual,” the part that cannot be explained by what we measure.
  - Measurement is very difficult – think about aggregating all capital into a single number for a country, and then imagine doing that for all countries!
  - To the extent things are mis-measured, the residual will grow

Causes of Growth

Education



Source: Mankiw et al (1992)

R&D

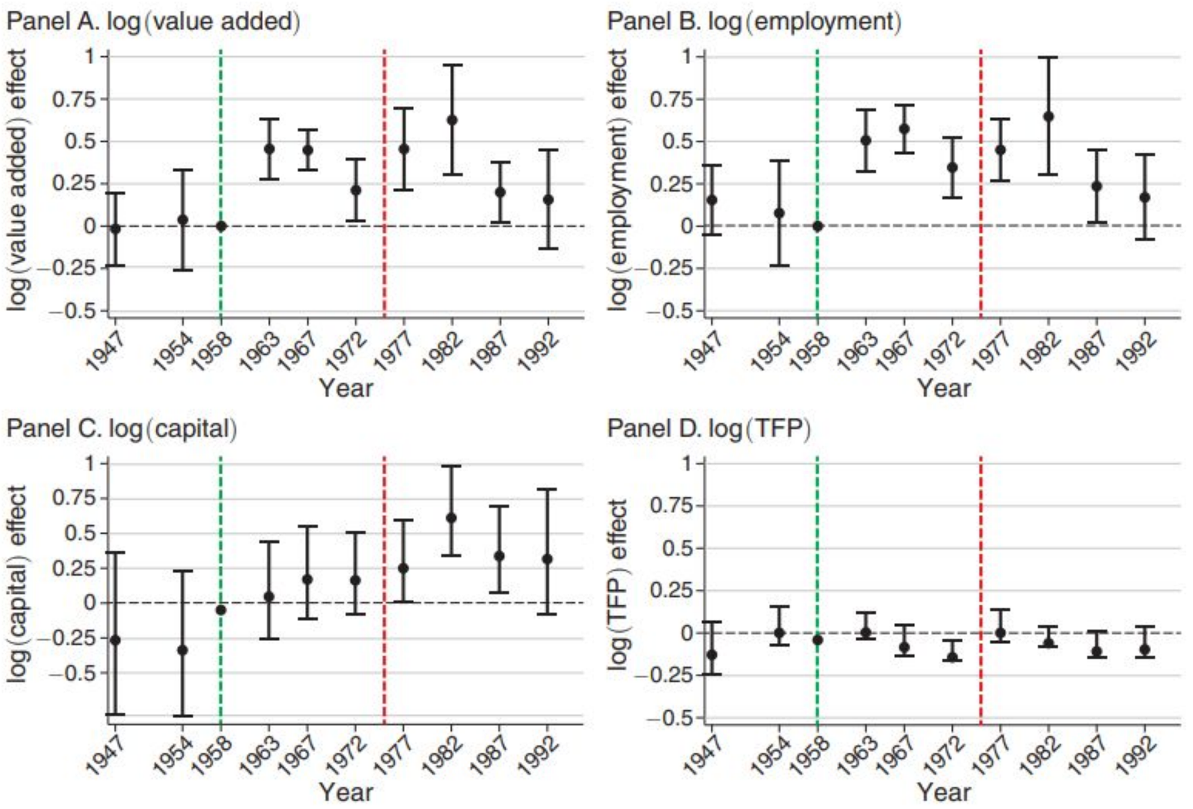
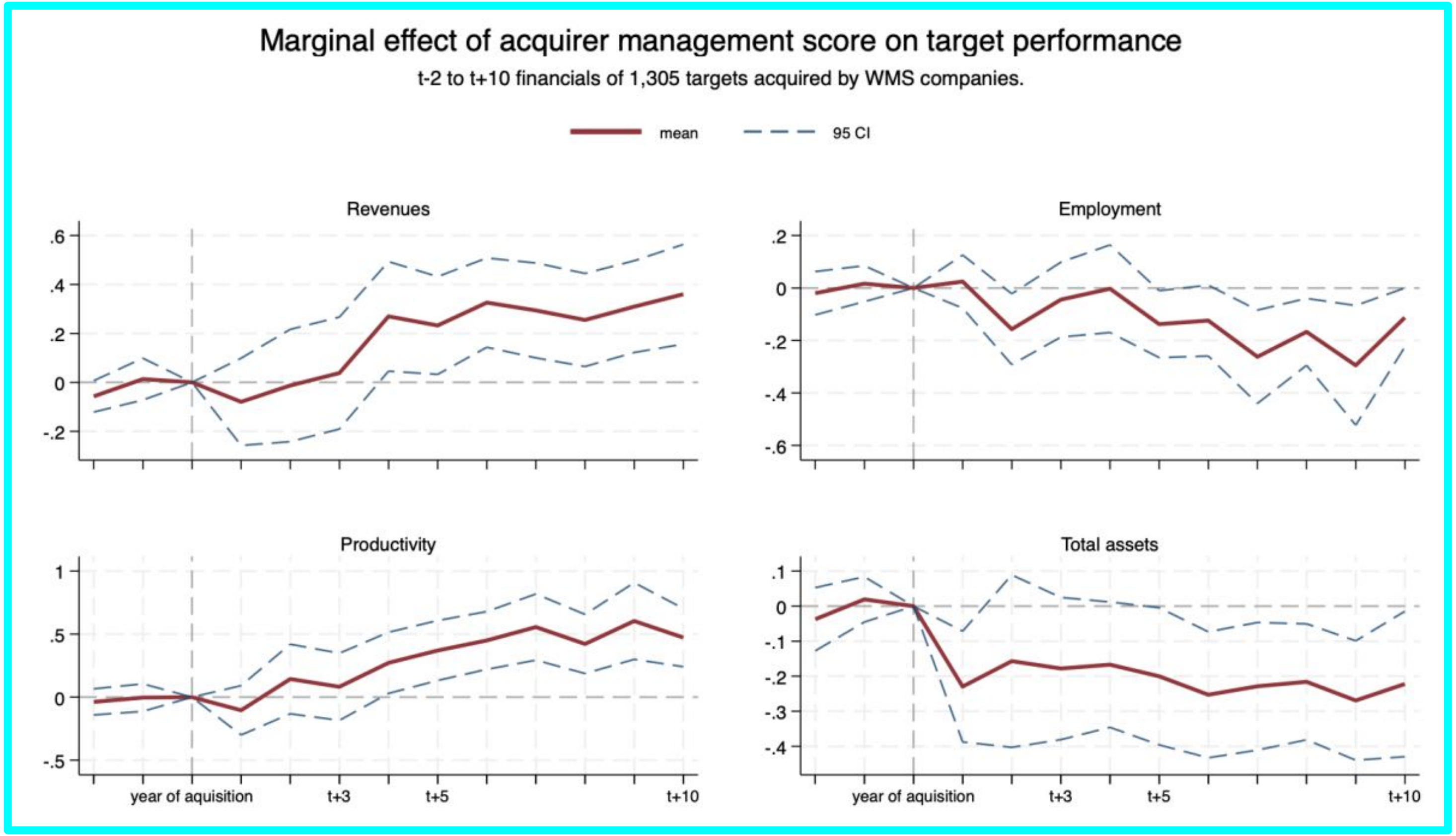


Figure 5. Space Capability and Manufacturing: Space Industry Effect Dynamics

Source: Kantor and Whalley (2025)

Management



Source: Bloom et al (2025)

Innovation, Competition, and Goldilocks

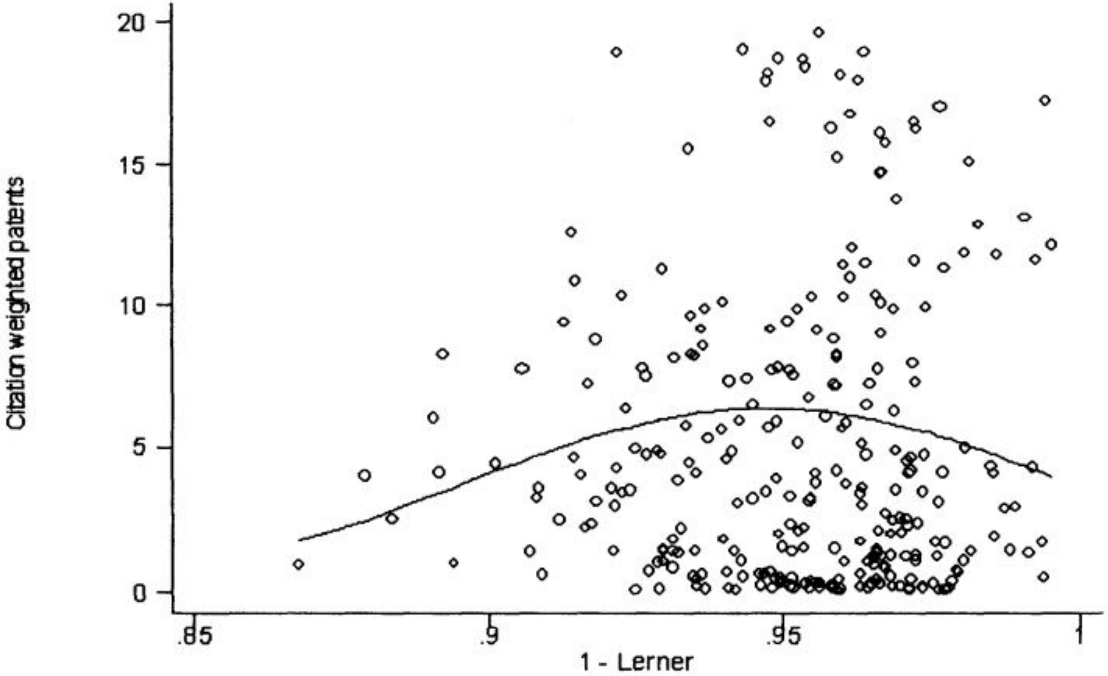


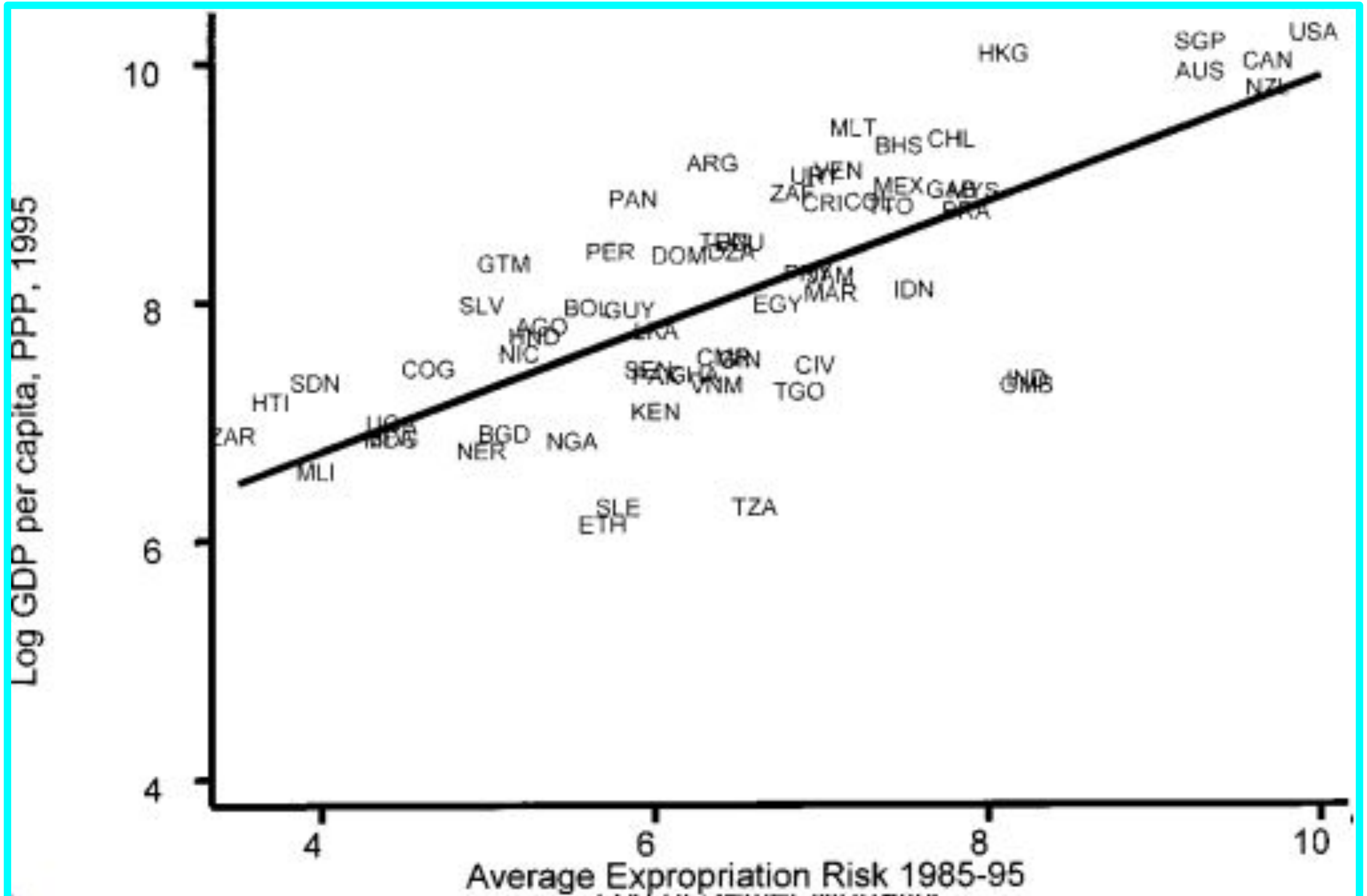
FIGURE I

Scatter Plot of Innovation on Competition

The figure plots a measure of competition on the x-axis against citation-weighted patents on the y-axis. Each point represents an industry-year. The scatter shows all data points that lie in between the tenth and ninetieth deciles in the citation-weighted patents distribution. The exponential quadratic curve that is overlaid is reported in column (2) of Table I.

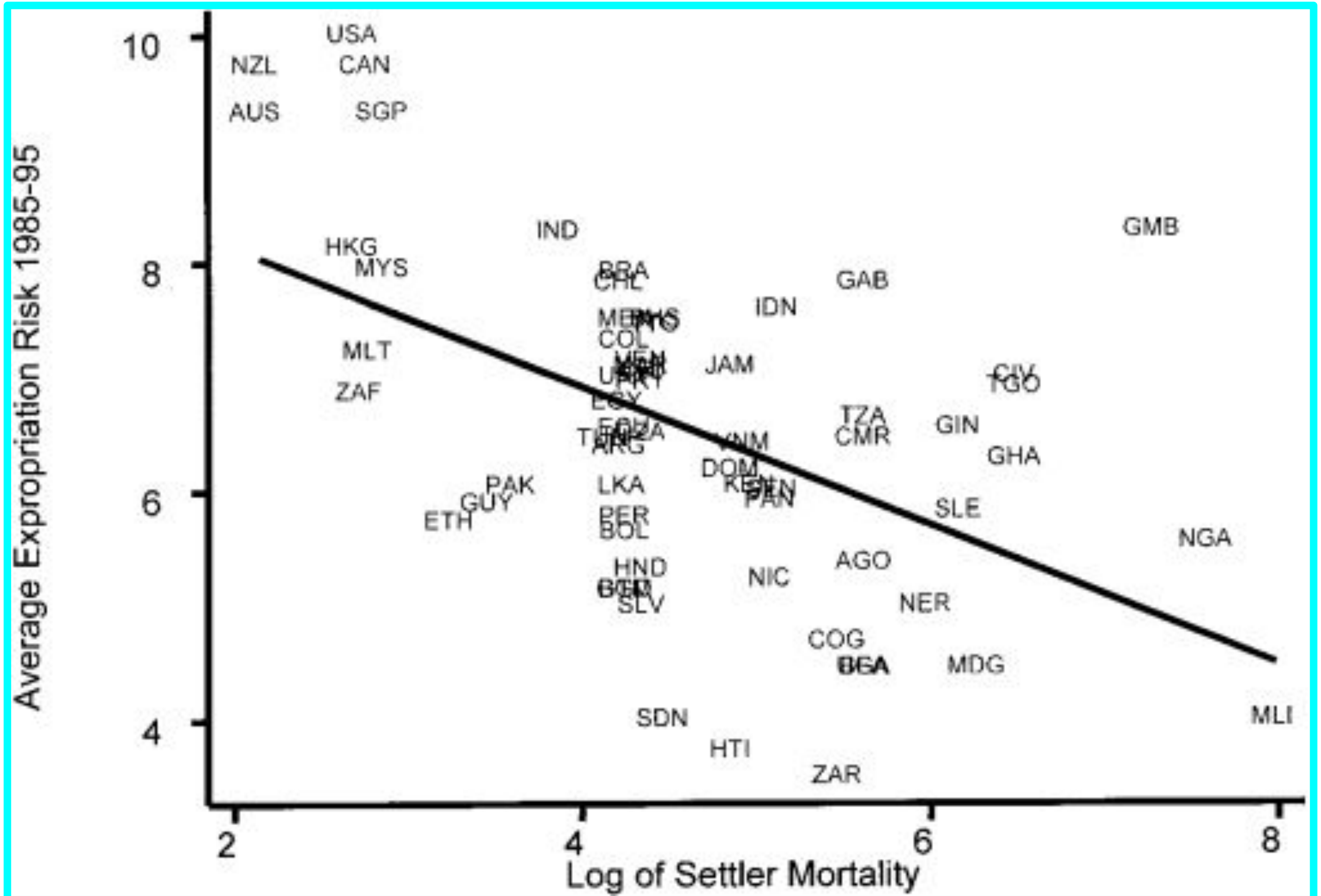
Source: Aghion et al (2005)

Political Institutions



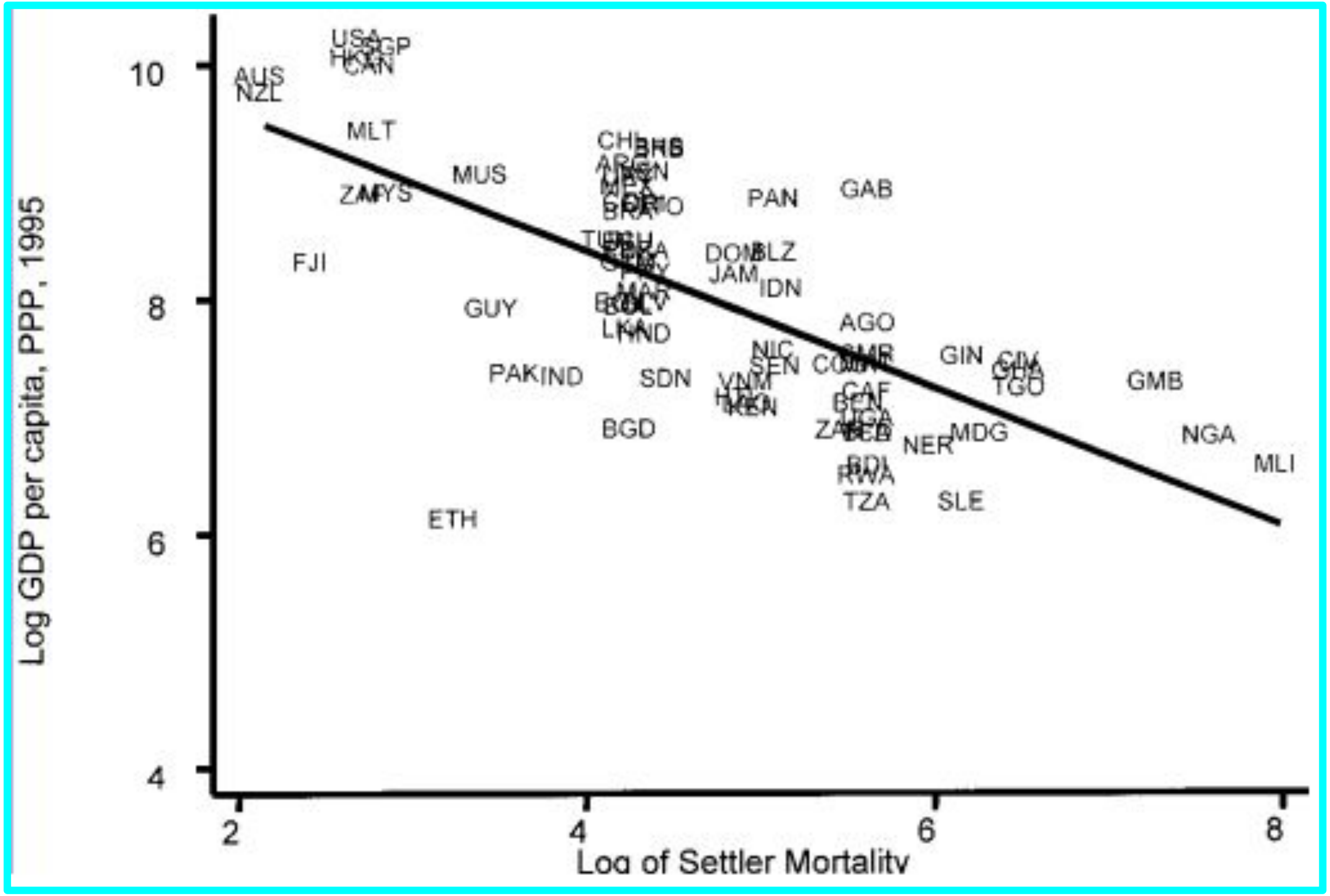
Source: Acemoglu et al (2001)

Colonial History



Source: Acemoglu et al (2001)

Colonial History (2)



Source: Acemoglu et al (2001)