Calculus Prerequisites

Econ 6105, Fall 2024

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(SB Chapters 2-5, 30.2, A.4, 14, 17)

Rate of Change

Consider function $f(x)$

- $f : R^1 \to R^1$, i.e. input x is scalar, output is a scalar

Consider two points in f's domain: x_0 and $x_0 + h$
Average rate of change of f from x_0 to $x_0 + h$ is:

$$\Delta_f(x_0; h) \equiv \frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0} = \frac{f(x_0 + h) - f(x_0)}{h} \tag{1}$$

- "Rise over run"
- "Within $[x_0, x_0 + h]$, what is the average increase in $f(x)$ when x increases by 1 unit?"

Instantaneous Rate of Change

$$\Delta_f(x_0; h) = \frac{f(x_0 + h) - f(x_0)}{h}$$

Seems reasonable to ask, "How quickly is f changing precisely at x_0?"
Tempting to set $h = 0$, but then we get indeterminate answer:

$$\Delta_f(x_0; 0) = \frac{f(x_0) - f(x_0)}{0} = 0/0 =??$$

Instead, we take the limit: $\lim_{h \to 0} \Delta_f(x_0, h)$

- "What is the average increase in $f(x)$ as x moves away from x_0, but by an arbitrarily small amount?"

This is known as the "derivative", denoted $f'(x)$ or df/dx.

Limit

A function, $f(x)$ has a limit of L as x approaches p if:

- For every $\epsilon > 0$...
- there exists a $\delta(\epsilon) > 0$ such that...
- if $|x - p| \in (0, \delta(\epsilon))$...
- then $|f(x) - L| < \epsilon$.

In notation:
$\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ s.t. $|x - p| \in (0, \delta(\epsilon)) \Rightarrow |f(x) - L| < \epsilon$

The Derivative

$$f'(x) \equiv \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad (2)$$

Example: $f(x) = x^2$, $x_0 = 3$

$$\begin{aligned}
f'(x_0) &\equiv \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
&= \lim_{h \to 0} \frac{(x_0 + h)^2 - x_0^2}{h} \\
&= \lim_{h \to 0} \frac{x_0^2 + h^2 + 2 \cdot x_0 \cdot h - x_0^2}{h} \\
&= \lim_{h \to 0} h + 2 \cdot x_0 \\
&= 2 \cdot x_0
\end{aligned}$$

$f'(3) = 6$.

Undefined Derivatives

Sometimes you get "different limits" when $h \to 0$ in different ways.

- More precisely, this means the limit does not exist

Consider $f(x) = |x|$. Suppose we have h approach 0 from above:

$$f'(0) = \frac{|0+h| - |0|}{h} = \frac{h}{h} = 1?$$

Now have h approach 0 from below:

$$f'(0) = \frac{|0+h| - |0|}{h} = \frac{-h}{h} = -1?$$

The derivative of $f(x) = |x|$ is not defined at $x = 0$ because the rate of change depends on which direction you're going.

We typically work with "well-behaved" functions, but you need to be careful when working on problems with sharp/discrete changes.

- E.g. Minimizing squared errors vs absolute errors

Special Functions

Some important cases to know cold:

- $f(x) = x^n \to f'(x) = n \cdot x^{n-1}$
- $f(x) = \ln(x) \to f'(x) = 1/x$
- $f(x) = a^x \to f'(x) = \ln(a) \cdot a^x$
  - $f(x) = e^x \to f'(x) = e^x$

Rules of Derivatives

1. For a scalar k: $(k \cdot f)'(x) = k \cdot f'(x)$
2. For two functions f and g: $(f + g)'(x) = f'(x) + g'(x)$
3. Product Rule: $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
4. Quotient Rule: $(f/g)'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x)^2}$
    - "Ho-dee-hi minus hi-dee-ho, all over hoho"

The Chain Rule

Consider $f(x) \equiv h(g(x))$.

- E.g. $g(x) = \ln(x)$, $h(x) = x^2 \rightarrow f(x) = \ln(x)^2$

Derivative of $f(x)$ is found with the Chain Rule:

$$\begin{aligned} f'(x_0) &= g'(x_0) \cdot h'(g(x_0)) \\ df/dx &= \frac{dg}{dx} \cdot \frac{dh}{dg} \end{aligned} \tag{3}$$

"x moves g by g', which then moves f by h' per unit: $f' = g' \cdot h'$."

- $f(x) = \ln(x)^2 \rightarrow f'(x) = 2 \cdot \ln(x) \cdot \frac{1}{x}$

The Chain Rule could just be "rule 5" on the previous slide, but it's a bit harder and quite important.

Higher-Order Derivatives

Can take the derivative of a derivative ("second derivative")...and derivative of second derivative ("third derivative), and so on...

- $f(x) = 1/x$

1. $f'(x) = -1/x^2$
2. $f''(x) = 2/x^3$
3. $f^{[3]}(x) = -6/x^4$
4. ...

Away from $x = 0$, all ∞ derivative functions exist. Therefore, $f(x) = 1/x$ is "continuously differentiable" – or "smooth" – for $x \neq 0$.

First-Order Taylor Approximation

A first order Taylor Polynomial around a in the domain of f is:

$$\tilde{f}(a+h) = f(a) + f'(a) \cdot h \tag{4}$$

Note that $\tilde{f}$ is a good approximation of f around a in the following sense:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(a+h) - \tilde{f}(a+h)}{h} &= \lim_{h \to 0} \frac{f(a+h) - (f(a) + f'(a) \cdot h)}{h} \\
&= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} - f'(a) \\
&= f'(a) - f'(a) \\
&= 0
\end{aligned}$$

In words, no matter how small your desired margin of error, you can find h small enough to meet it.

Higher-Order Taylor Approximations

Common to see second-order Taylor approximation:

$$\tilde{f}(a+h) \approx f(a) + f'(a)\cdot h + \frac{1}{2}\cdot f''(a)\cdot h^2 \tag{5}$$

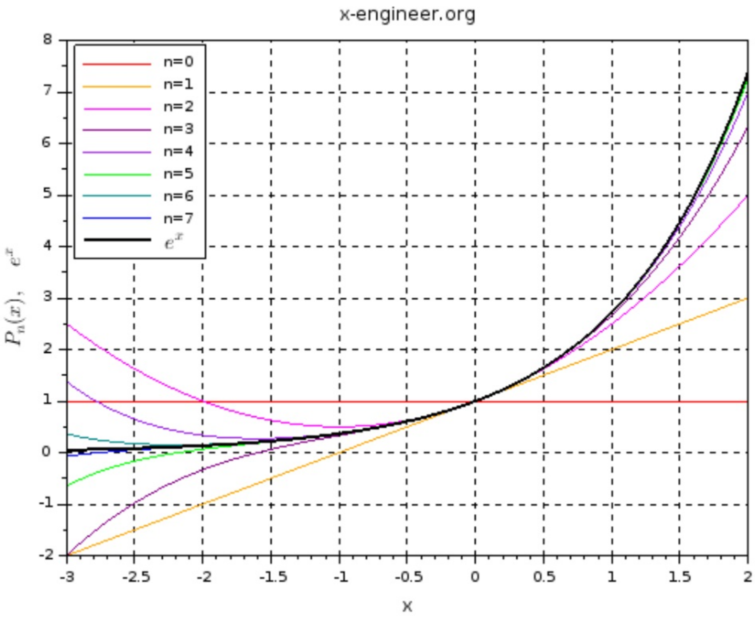
Won't prove this one, but intuition is that both first and second derivatives are correct at a:

- $\lim_{h\to 0} \tilde{f}'(a+h) = \lim_{h\to 0} f'(a) + f''(a)\cdot h = f'(a)$
- $\lim_{h\to 0} \tilde{f}''(a+h) = \lim_{h\to 0} f''(a) = f''(a)$

Can do higher-order approximations:

$$\tilde{f}(a+h) \approx f(a)+f'(a)\cdot h+\frac{1}{2}\cdot f''(a)\cdot h^2+\frac{1}{2\cdot 3}\cdot f^{(3)}(a)\cdot h^3+\ldots+\frac{1}{n!}\cdot f^{(n)}(a)\cdot h^n+\ldots \tag{6}$$

Taylor Approximations (picture)



Concavity/Convexity

First and second derivatives are often discussed.

- f' determines increasing (> 0) vs. decreasing (< 0)
- f'' determines convex (> 0) vs. concave (< 0)

Cases:

- $f'(x) > 0$: $f(x)$ is increasing...
  1. $f''(x) > 0$: ...at an increasing rate
  2. $f''(x) < 0$: ...at a decreasing rate
- $f'(x) < 0$: $f(x)$ is decreasing...
  1. $f''(x) > 0$: ...at an increasing rate
  2. $f''(x) < 0$: ...at a decreasing rate

The Maximum of a Function

Want to find the maximum of a function $f(x)$ on the domain $[a, b]$.

1. Can rule out points with $f'(x) \neq 0$. Why?
2. Can rule out points $f'(x) = 0$ and $f''(x) > 0$. Why?

What's left?

1. Points with $f'(x) = 0$ and $f''(x) < 0$ – local maxima
2. Points with $f'(x) = 0$ and $f''(x) = 0$ – possible local maxima...
3. Points where $f'(x)$ or $f''(x)$ are not defined
    - Most notably, the boundaries

Unconstrained Maximization Cookbook

General approach

1. Calculate $f'(x)$
2. Identify points $\{x_1, x_2, ...\}$ with $f'(x) = 0$ or $f'(x)$ undefined
3. Calculate $f(x)$ at all such points – choose the largest.

In many economics settings, we work with functions such that $f'(x) > 0$ and $f''(x) < 0$ for all x

- Increasing, concave functions

In this case, there is exactly 1 local maximum, and it is the global maximum.

- Find it by setting $f'(x) = 0$ and solving for x.

The Antiderivative

Define $F(x)$ to be the "antiderivative" of $f(x)$, meaning $F'(x) = f(x)$.
Just uses rules of derivatives in reverse

- E.g. If $f(x) = 2 \cdot x$, then $F(x) = x^2 + C$, where C is a scalar constant
  - It works: $F'(x) = 2 \cdot x + 0$
  - C is called the "constant of integration"

The antiderivative is not unique: any value of C would work, so there are ∞ solutions.
Anitderivative (also called the "indefinite integral") of $f(x)$ is helpful for calculating "area under a curve"...

Integration

Derivatives measure an instantaneous change

- E.g. "How much water is flowing into the tub at this instant?"

Integrals measure an accumulated change

- E.g. "How much water flowed into the tub between times a and b?"

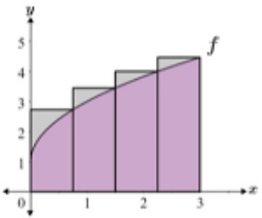
Let $f(x)$ be the instantaneous flow of water, defined for $x \in [a, b]$.
Define W to be amount of water accumulated from $x = a$ to $x = b$.
W can be:

1. Approximated as a sum
2. Solved exactly as a "definite integral"

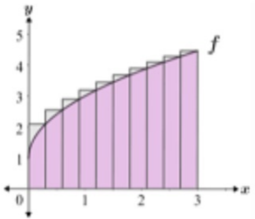
Riemann Sum

To calculate Riemann Sum of area under curve:

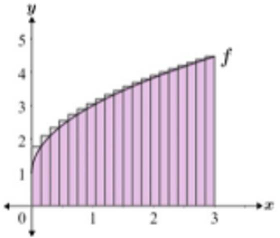
1. Partition $[a, b]$ into intervals
   - For simplicity, $N \equiv \frac{b-a}{\Delta}$ equally-sized intervals of width Δ, indexed by i
2. Assign height to each section, $\tilde{f}(x)_i$, and create corresponding rectangle
   - In picture, height is $\tilde{f}(x)_i = f(\max\{x\})$, i.e. $f(x)$ at rightmost point in each interval
3. Calculate rectangle areas and sum them up



4 circumscribed rectangles



10 circumscribed rectangles



20 circumscribed rectangles

Source: Math.net

The Fundamental Theorem of Calculus

Fundamental Theorem of Calculus:

$$W = \lim_{\Delta \to 0} \sum_{i=1}^{N} \tilde{f}(x)_i \cdot \Delta \equiv \int_a^b f(x) \cdot dx \equiv F(b) - F(a). \tag{7}$$

E.g. $f(x) = 2 \cdot x$, $a = 0$, $b = 4$:

- $F(x) = x^2 + C$
- $\int_a^b f(x) \cdot dt = 4^2 + C - (0^2 + C) = 16$

Does not matter that we never solved for C – it canceled out!

- C is like an initial condition at time a
- We do not need to know how much water was in the tub at time a to know how much flowed in between a and b.

Multivariate Functions

We will now consider multivariate functions, $f : R^n \rightarrow R^1$

- E.g. Utility depends on multiple goods $u(x_1, x_2) = x_1^{\alpha} \cdot x_2^{1-\alpha}$
- E.g. $H(x, t)$ heat depends on time of day (t) as well as actions taken by agent (x).
    - Profit may depend on price, and quatity – which depends on price: $\Pi(x(p), p)$

Multivariate calculus builds very directly off of single-variable calculus

Partial Derivative

For $f(x) \equiv f(x_1, ..., x_n)$, the "partial derivative of f with respect to x_i" is the impact of a marginal change in x_i, holding all other $x_{j \neq i}$ constant:

$$\frac{\partial f}{\partial x_i}(x^0) = \lim_{h \to 0} \frac{f(x_1^0, ...x_i^0 + h, ..., x_n^0) - f(x_1^0, ...x_i^0, ..., x_n^0)}{h} \quad (8)$$

This essentially identical to the derivative from the single-variate case ("∂" instead of "d").

The multivariate analog of "the derivative" (called the "Jacobian" or "gradient") of f is just the collection the individual partial derivatives:

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right].$$

Total Derivative

Partial derivative moves x_i, holds all $x_{j \neq i}$ constant. In contrast, "total derivative" considers simultaneous marginal changes in all variables. In particular, for small dx_is:

$$df(x^0) = \frac{\partial f}{\partial x_1}(x^0) \cdot dx_1 + ... + \frac{\partial f}{\partial x_n}(x^0) \cdot dx_n \tag{9}$$

E.g. Change in utility from perturbing bundle (x_1, x_2) is:

$$dU = \frac{\partial U}{\partial x_1} \cdot dx_1 + \frac{\partial U}{\partial x_2} \cdot dx_2 = MU_1 \cdot dx_1 + MU_2 \cdot dx_2.$$

Holding utility constant, we get the slope of the indifference curve:

$$\frac{dx_2}{dx_1}|_{dU=0} = -\frac{MU_1}{MU_2}$$

Rules of Derivatives in the Multivariate Case

All of the rules discussed earlier for single-variable derivatives carry over for partial derivatives.

Note that Chain Rule can become more interesting in the multivariate case. Suppose $f(Y) = g(x_1(Y), ..., x_n(Y))$. Then:

$$\frac{df}{dY}(Y^0) = \frac{\partial g}{\partial x_1}(x(Y^0)) \cdot x_1'(Y^0) + ... + \frac{\partial g}{\partial x_n}(x(Y^0)) \cdot x_n'(Y^0)$$

To get the impact of income (Y) on f...

1. Look at how each x_i is affected ($x_i'(Y^0)$)
2. Multiply by the sensitivity of g to that particular x_i ($\frac{\partial g}{\partial x_i}$)
3. Sum up across all i

Interesting Special Case: Leibniz Rule

Let $W(x) = \int_{a(x)}^{b(x)} f(x,t) \cdot dt$. Then:

$$\frac{dW}{dx} = \underbrace{f(x, b(x)) \cdot \frac{db}{dx}}_{\text{Gain on margin}} - \underbrace{f(x, a(x)) \cdot \frac{da}{dx}}_{\text{Lose on margin}} + \underbrace{\int_{a(x)}^{b(x)} \frac{\partial f}{\partial x} \cdot dt}_{\text{Inframarginal changes}} \tag{10}$$

Second Derivatives of Multivariate Functions

There are two types of "second partial derivatives":

1. "Own second": $f_{x_i x_i} \equiv \frac{\partial^2 f}{\partial x_i^2}$
2. "Cross partial": $f_{x_i x_j} \equiv \frac{\partial^2 f}{\partial x_i \partial x_j}$

Hessian is a collection of all second derivatives

$$\mathbf{H}_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Source: Wikipedia

- Own partials form diagonal
- Cross partials are off-diagonal
  - Symmetry: $f_{x_i x_j} = f_{x_j x_i}$

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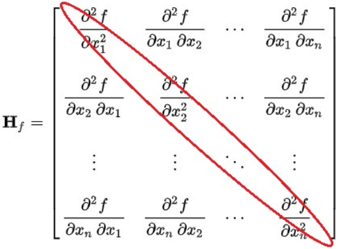

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Extrema of Multivariate Functions

As with single-variable functions, a "derivative" of 0 indicates a local extremum:

$$\nabla f(\mathbf{x}) = \mathbf{0}$$

i.e. $\frac{\partial f}{\partial x_i} = 0 \; \forall i$.
Again, in general need to worry about multiplicity of local extrema, boundaries, strange points.
But with an concave function, a local maximum will be unique, and it will be the global maximum.
A multivariate function is concave if its Hessian is "negative definite."

- We'll now need to discuss some linear algebra.